

The calculation of energies and properties of large systems

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## The one-electron density matrix

- Traditional SCF theory rests on the concept of molecular orbitals (MOs):

$$\mathbf{FC} = \mathbf{SC}\epsilon \leftarrow \text{Roothaan-Hall equations}$$

- cost of diagonalization is cubic—expensive for large systems
- ill suited to parallelization

- Fortunately, MOs are not needed: all information is contained in the density matrix

$$\mathbf{D} = \mathbf{C}_{\text{occ}}^{\text{T}} \mathbf{C}_{\text{occ}} \leftarrow \text{density matrix}$$

- it is sufficient to work in terms of the one-electron density matrix
- this is particularly advantageous if it can be compactly (sparsely) represented

- This approach is taken in our work, but then several questions arise:

- Q1: In what basis do we represent the density matrix?
- Q2: How do we optimize the density matrix, without recourse to orbitals?
- Q3: How do we calculate its response to perturbations?

- All tasks should be accomplished in terms of elementary matrix manipulations

- additions, multiplications, trace operations—no diagonalizations
- with sparsity, linear scaling is then achievable for large systems
- diagonalization-free methods are simpler to parallelize

## Overview

- the generation of a sparse, orthonormalized atomic-orbital (OAO) basis
  - diagonalization-free evaluation of  $\mathbf{S}^{\pm 1/2}$
  - multiplication-rich globally convergent iterative method
  - Jansik *et al.*, *J. Chem. Phys.* **126**, 124104 (2007)
- energy optimization for large systems
  - direct optimization of density matrix  $\mathbf{D}$  in OAO basis
  - multiplication-rich diagonalization-free self-consistent field (SCF) theory
  - Salek *et al.*, *J. Chem. Phys.* **126**, 114110 (2007)
- linear response theory for large systems
  - direct perturbation of density matrix  $\mathbf{D}$  in OAO basis
  - calculation of polarizabilities and excitation energies
  - Coriani *et al.* *J. Chem. Phys.* **126**, 154108 (2007)

## Part I: Orthonormalization of the AO basis

- For large systems, the **atomic-orbital (AO) basis** is ideal in the sense that it is sparse
  - unfortunately, many computational tasks are ill-conditioned in this basis
- This may be avoided by transforming to an **orthonormalized AO (OAO) basis**

$$\mathbf{Z}^T \mathbf{S} \mathbf{Z} = \mathbf{I}, \quad \mathbf{Z} = \begin{cases} \mathbf{S}^{-1/2} & \text{Löwdin} \\ \mathbf{U}^{-1} \text{ (upper triangular)} & \text{Cholesky} \end{cases}$$

- Millam and Scuseria, JCP **106**, 5569 (1997); Challacombe, JCP **110**, 2332 (1999)
- The **Löwdin basis** resembles the AO basis most closely (Carlson and Keller, 1957)
  - it should therefore retain sparsity to the greatest possible extent
  - however, its construction traditionally involves diagonalization
- The **Cholesky basis** has therefore been preferred instead
  - many uses in quantum chemistry (rank reduction techniques)
- However, since the Löwdin basis is (presumably) the sparsest of all OAO bases, it would be surprising if it could not be generated in linear time!
- We have recently presented a robust linear-scaling algorithm for Löwdin factorization
  - Jansik *et al.*, J. Chem. Phys. **126**, 124104 (2007)

## A globally convergent Newton–Schulz method for the inverse square root

- Application of Newton’s elementary method for root finding gives

$$Z^{-2} - S = 0 \quad \Rightarrow \quad Z_{n+1} = \frac{3}{2}Z_n - \frac{1}{2}Z_n^3 S \quad \leftarrow \text{Newton–Schulz iteration}$$

- $Z_n$  converges quadratically but not globally to  $S^{-1/2}$ ; no inversion!

- In 2004, Niklasson proposed the corresponding matrix iteration

$$\mathbf{Z}_{n+1} = \frac{1}{2}\mathbf{Z}_n \left( 3\mathbf{I} - \mathbf{Z}_n^T \mathbf{S} \mathbf{Z}_n \right) \quad \leftarrow \text{stabilized Newton–Schulz matrix iteration}$$

- higher-order formulas may also be derived
- for  $\mathbf{Z}_0 = \mathbf{I}$ , there are severe convergence restrictions  $\|\mathbf{S} - \mathbf{I}\|_2 < 1$  for such methods
- intended for refinement of approximate inverse square roots

- However, convergence is guaranteed with the scaled  $\lambda_{\min} \mathbf{S}$  that minimizes  $\|\lambda \mathbf{S} - \mathbf{I}\|_2$ :

$$\lambda_{\min} = \frac{2}{\epsilon_{\min} + \epsilon_{\max}} \quad \leftarrow \text{expensive}$$

- to avoid eigenvalues, we replace the 2-norm by a ratio of Frobenius norms

$$f(\lambda) = \sqrt{\frac{\text{Tr}(\lambda \mathbf{S} - \mathbf{I})^4}{\text{Tr}(\lambda \mathbf{S} - \mathbf{I})^2}} \quad \leftarrow \text{cheap lower bound to the 2-norm}$$

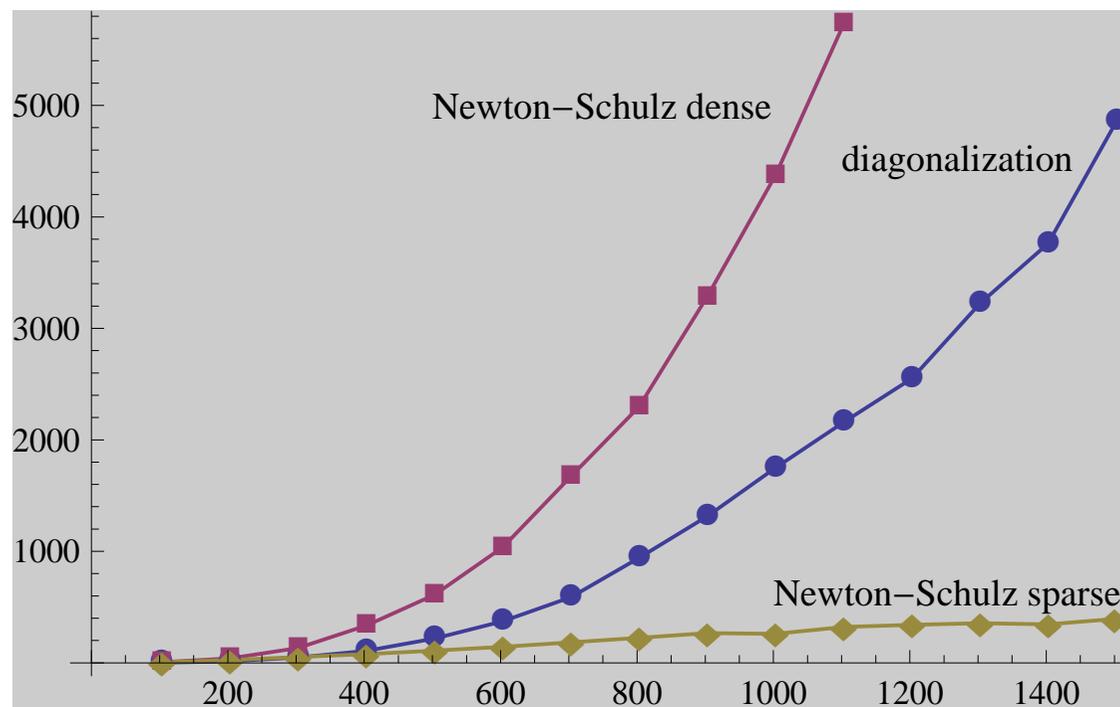
- the evaluation and minimization of  $f(\lambda)$  costs only **one** matrix multiplication

Number of iterations for convergence to  $10^{-10}$

	$\ \mathbf{S} - \mathbf{I}\ _2$	$N^{(2)}$	$N^{(3)}$
Fullerene $\text{C}_{60}$	6.0	17	11
Polysaccharide (438)	5.2	14	10
Polysaccharide (875)	5.2	14	10
DNA fragment (583)	4.9	15	11
PRC model (1082)	5.7	16	11
Red phosphorus (506)	1.9	8	6
Red phosphorus (810)	4.7	14	10
Crystobalite (984)	4.4	13	9
Tridymite (1314)	5.2	15	11
Random molecule (64)	16.0	23	27

## Performance of the scaled Newton–Schulz method

- About 15 scaled Newton–Schulz iterations needed for convergence to  $10^{-10}$ 
  - less than 50 matrix multiplies
  - example: polyalanine peptide residues (6-31G) (cost vs. number of atoms)
- In dense-matrix algebra:
  - of cubic complexity
  - two–three times more expensive than diagonalization
  - easy to parallelize
  - beats diagonalization on 4–6 processors
- In sparse-matrix algebra:
  - of linear complexity
  - beats diagonalization for a few hundred atoms
- Transformation to orthonormal basis is an essential preparatory step to optimization



## Part II: Energy optimization

- In traditional SCF theory, we solve the **Roothaan–Hall (RH) eigenvalue problem**

$$\mathbf{FC} = \mathbf{SC}\epsilon, \quad \mathbf{D}_{\text{new}} = \mathbf{C}_{\text{occ}}\mathbf{C}_{\text{occ}}^{\text{T}}$$

- Its solution is equivalent to minimizing the sum of the occupied orbital energies

$$\epsilon_{\text{min}} = \min \sum_i \epsilon_i = \min_{\mathbf{D}} \text{Tr } \mathbf{D}\mathbf{F} \leftarrow \text{not all } \mathbf{D} \text{ variations valid!}$$

- The valid variations may be represented by an antisymmetric matrix  $\mathbf{X}$ :

$$\mathbf{D}(\mathbf{X}) = \exp(-\mathbf{X}\mathbf{S})\mathbf{D}\exp(\mathbf{S}\mathbf{X}), \quad \mathbf{X}^{\text{T}} = -\mathbf{X} \leftarrow \text{exponential parametrization}$$

- these transformations satisfy the symmetry, trace and idempotency conditions

$$\mathbf{D}(\mathbf{X})^{\text{T}} = \mathbf{D}(\mathbf{X}), \quad \text{Tr } \mathbf{D}(\mathbf{X})\mathbf{S} = N_{\text{el}}, \quad \mathbf{D}(\mathbf{X})\mathbf{S}\mathbf{D}(\mathbf{X}) = \mathbf{D}(\mathbf{X})$$

- Helgaker *et al.*, CPL **327**, 397 (2000); Head-Gordon *et al.*, MP **101**, 37 (2003)

- Each RH diagonalization is therefore equivalent to the **trace minimization problem**

$$\epsilon_{\text{min}} = \min_{\mathbf{X}} \text{Tr } \mathbf{D}(\mathbf{X})\mathbf{F}, \quad \mathbf{D}_{\text{new}} = \mathbf{D}(\mathbf{X}_{\text{min}})$$

- minimization performed with Newton's method

## The Roothaan–Hall Newton method

- To carry out the trace minimization, we expand the energy in powers of  $\mathbf{X}$ :

$$\begin{aligned}\text{Tr } \mathbf{F} \mathbf{D}(\mathbf{X}) &= \text{Tr } \mathbf{F} \exp(-\mathbf{X}) \mathbf{D} \exp(\mathbf{X}) \\ &= \text{Tr } \mathbf{F} \mathbf{D} + \text{Tr } \mathbf{F} [\mathbf{D}, \mathbf{X}] + \frac{1}{2} \text{Tr } \mathbf{F} [[\mathbf{D}, \mathbf{X}], \mathbf{X}] + \dots\end{aligned}$$

- Truncating at second order and setting the gradient to zero, we obtain the Newton step:

$$(\mathbf{F}^{\text{vv}} - \mathbf{F}^{\text{oo}}) \mathbf{X} + \mathbf{X} (\mathbf{F}^{\text{vv}} - \mathbf{F}^{\text{oo}}) = \mathbf{F}^{\text{vo}} - \mathbf{F}^{\text{ov}} \leftarrow \text{Roothaan–Hall Newton equations}$$

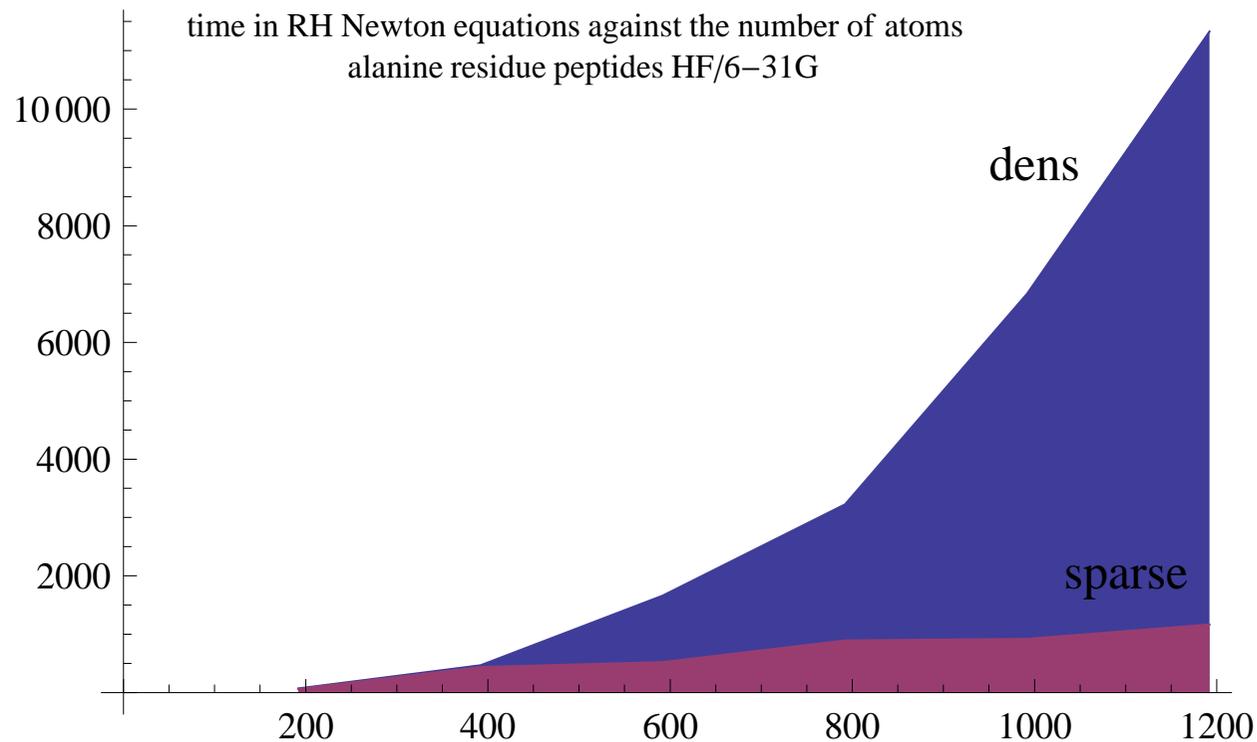
- upper indices indicate projections onto the occupied and virtual spaces

$$\mathbf{F} = \mathbf{F}^{\text{oo}} + \mathbf{F}^{\text{ov}} + \mathbf{F}^{\text{vo}} + \mathbf{F}^{\text{vv}}, \quad (\mathbf{P}_o = \mathbf{D}, \mathbf{P}_v = \mathbf{I} - \mathbf{D})$$

- Because of their large dimensions, the Newton equations cannot be solved directly
  - solution by **the preconditioned conjugate-gradient method** (typically 10 iterations)
  - elementary (sparse) matrix manipulations (typically less than 100 multiplications)
  - use of an orthonormal basis essential for rapid convergence
- A RH diagonalization corresponds to an exact minimization (many Newton steps)
  - however, a **partial minimization** will do (one RH Newton step is sufficient)
- Larsen *et al.*, J. Chem. Phys. **113** 8908 (2000); Shao *et al.*, *ibid.* **118** 6144 (2003)  
Sialek *et al.*, *ibid.* **126** 114110 (2007)

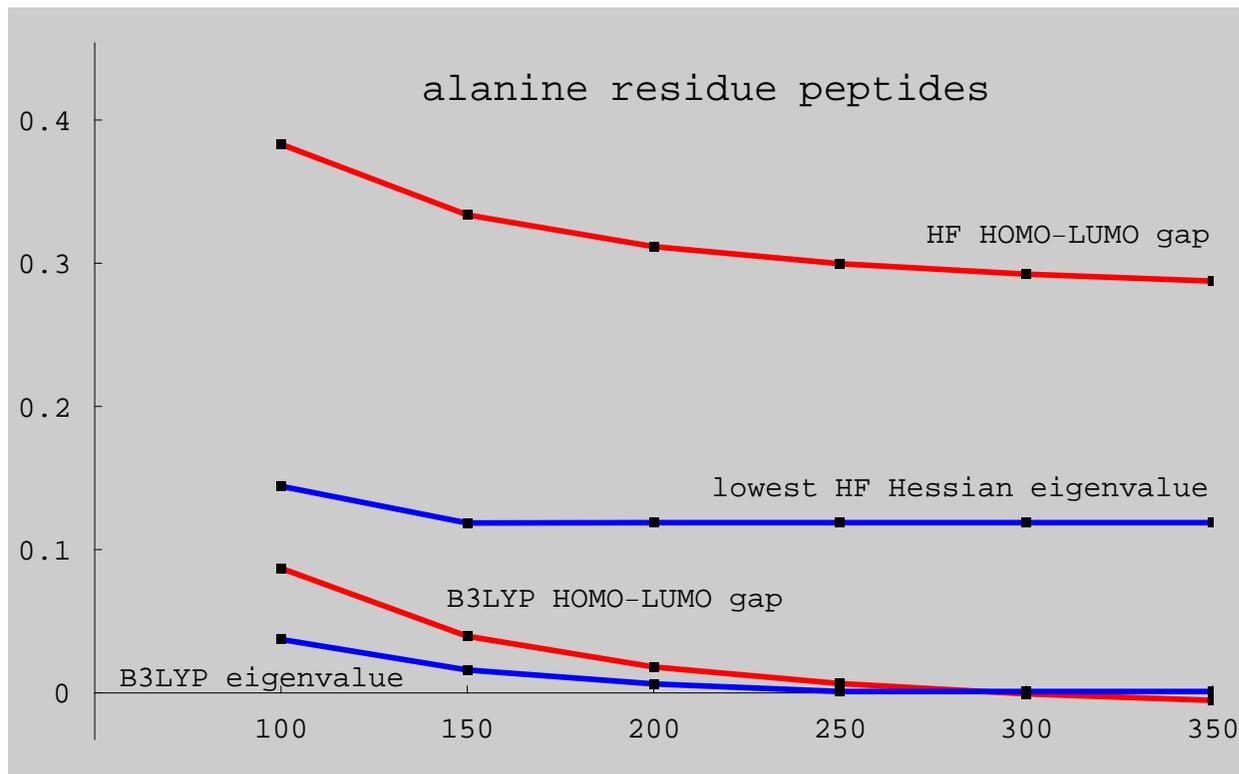
## Performance of the Roothaan–Hall Newton method

- We have successfully avoided Roothaan–Hall diagonalization
  - minimization rather than the solution of a generalized eigenvalue problem
  - rapidly convergent: 50–100 sparse matrix multiplications needed
  - well suited to parallelization
- Linear scaling is obtained by employing sparse-matrix algebra
  - compressed sparse-row (CSR) representation of few-atom blocks



## SCF optimizations in small and large molecules

- Diagonalization can be avoided by solving Newton equations
- However, SCF convergence is typically more difficult in larger systems
  - small (or negative) HOMO–LUMO gaps and small Hessian eigenvalues in DFT
  - 6-31G HOMO–LUMO gap and lowest Hessian eigenvalue in alanine residue peptides



- The standard SCF scheme must be modified, making it more robust.

## The trust-region self-consistent field (TRSCF) method

- SCF optimizations have two ingredients
  1. Roothaan–Hall minimization (diagonalization):  $\min_{\mathbf{X}} \text{Tr } \mathbf{D}(\mathbf{X})\mathbf{F}$
  2. DIIS-type averaging of density matrices:  $\bar{\mathbf{D}} = \sum_{i=0}^n c_i \mathbf{D}_i$ ,  $\sum_i c_i = 1$
- In the Roothaan–Hall step, we minimize subject to a constraint on the step size:
  - this amounts to a simple level shifting of the Fock/Kohn–Sham matrix

$$\mathbf{F} \rightarrow \mathbf{F}(\mu) = \mathbf{F} - \mu \mathbf{S}^{\text{oo}} \quad (\text{only occupied–occupied part shifted})$$

- for  $\mu > 0$ , the HOMO–LUMO gap increases, making large steps unfavourable
  - $\mu$  is adjusted until step is of desired length, during the iterative solution
- In the averaging step, we construct a second-order model of the SCF energy

$$E_{\text{DSM}}(c_i) \approx E_{\text{SCF}}(3\bar{\mathbf{D}}\mathbf{S}\bar{\mathbf{D}} - 2\bar{\mathbf{D}}\mathbf{S}\bar{\mathbf{D}}\mathbf{S}\bar{\mathbf{D}}), \quad \bar{\mathbf{D}} = \sum_i c_i \mathbf{D}_i$$

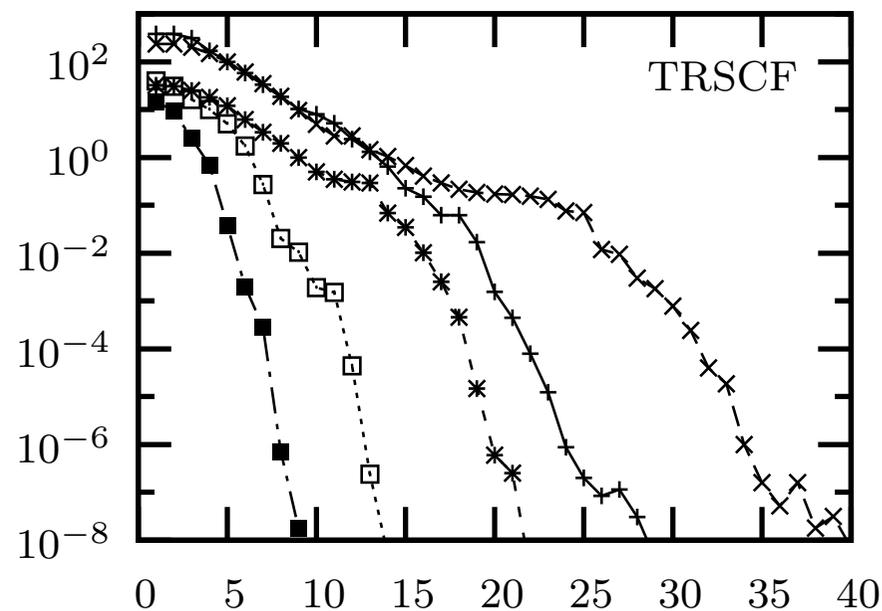
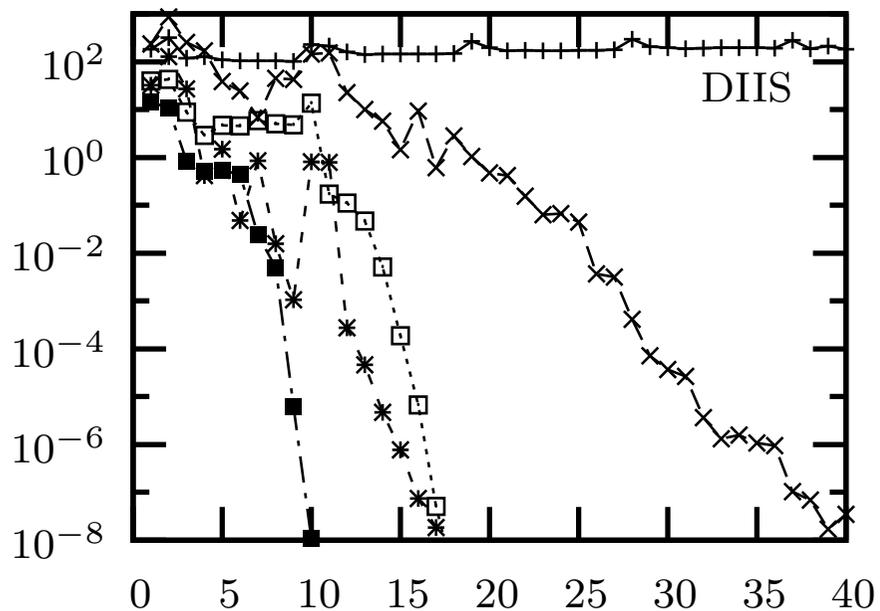
- it has the correct gradient but an approximate Hessian
  - the best density matrix is obtained by minimization, subject to a step-size constraint

$$\min_{c_i} E_{\text{DSM}}(c_i) \leftarrow \text{density-subspace minimization (DSM)}$$

- JCP **121**, 15 (2004); JCP **123**, 074103 (2005)

## The TRSCF method (continued)

- The TRSCF method gives a stable and uniform convergence towards the SCF minimum
- Convergence of LDA calculations for a variety of molecules
  - zinc complex(+), rhodium complex( $\times$ ), cadmium complex(\*),  $\text{CH}_3\text{CHO}$ ( $\square$ ) and  $\text{H}_2\text{O}$ ( $\blacksquare$ )



### Part III: Response theory

- We consider a system described by the one-electron density matrix

$$\mathbf{D}(\mathbf{X}) = \exp(-\mathbf{X})\mathbf{D}\exp(\mathbf{X}) \quad (\mathbf{X} = \mathbf{0} \text{ for unperturbed system})$$

and define the Hessian and metric operators in terms of their transformations

$$\mathbf{E}^{[2]}(\mathbf{X}) = (\mathbf{F}^{\text{vv}} - \mathbf{F}^{\text{oo}})\mathbf{X} + \mathbf{X}(\mathbf{F}^{\text{vv}} - \mathbf{F}^{\text{oo}}) + \mathbf{G}^{\text{vo}}([\mathbf{D}, \mathbf{X}]) - \mathbf{G}^{\text{ov}}([\mathbf{D}, \mathbf{X}])$$

$$\mathbf{S}^{[2]}(\mathbf{X}) = \mathbf{X}^{\text{ov}} - \mathbf{X}^{\text{vo}}$$

- When perturbed by  $\mathbf{V}_\omega$  of frequency  $\omega$ , the system responds to first order as

$$\mathbf{E}^{[2]}(\mathbf{X}_\omega) - \omega\mathbf{S}^{[2]}(\mathbf{X}_\omega) = [\mathbf{D}, \mathbf{V}_\omega] \quad \leftarrow \text{linear response matrix equation}$$

– perturbed density matrix:

$$\mathbf{D}_\omega = [\mathbf{D}, \mathbf{X}_\omega]$$

– perturbed expectation values:  $\langle\langle \hat{A}; \hat{V}_\omega \rangle\rangle_\omega = \text{Tr } \mathbf{A} [\mathbf{D}, \mathbf{X}_\omega]$

- In the absence of a perturbation  $\mathbf{V}_\omega = \mathbf{0}$ , we obtain an eigenvalue equation

$$\mathbf{E}^{[2]}(\mathbf{X}_n) = \omega_n\mathbf{S}^{[2]}(\mathbf{X}_n) \quad \leftarrow \text{RPA matrix eigenvalue equation}$$

– transition density matrix:

$$\mathbf{D}_{0n} = [\mathbf{D}, \mathbf{X}_n]$$

– transition moments:

$$\langle 0 | \hat{A} | n \rangle = \text{Tr } \mathbf{A} [\mathbf{D}, \mathbf{X}_n]$$

- Coriani *et al.*, J. Chem. Phys. **126**, 154108 (2007)

– Ochsenfeld, Head-Gordon, Weber, Niklasson, and Challacombe (static properties)

## Solution of the response equations

- For the solution, it is useful to consider two response equations

$$\mathbf{E}^{(2)}(\mathbf{X}) - \omega \mathbf{S}^{(2)}(\mathbf{X}) = \mathbf{B} \leftarrow \text{full response equation}$$

$$\mathbf{E}_{\text{F}}^{(2)}(\mathbf{X}) - \omega \mathbf{S}^{(2)}(\mathbf{X}) = \mathbf{B} \leftarrow \text{simplified response equation}$$

where the simplified Hessian is a good but cheap approximation to the full Hessian:

$$\mathbf{E}_{\text{F}}^{(2)}(\mathbf{X}) = (\mathbf{F}^{\text{vv}} - \mathbf{F}^{\text{oo}})\mathbf{X} + \mathbf{X}(\mathbf{F}^{\text{vv}} - \mathbf{F}^{\text{oo}}) \leftarrow \text{no two-electron part}$$

– its eigenvalues are the orbital energy differences  $\epsilon_a - \epsilon_i$

- We wish solve the full response equations iteratively

$$\mathbf{R}_i = \mathbf{E}^{(2)}(\mathbf{X}_i) - \omega \mathbf{S}^{(2)}(\mathbf{X}_i) - \mathbf{B} \leftarrow \text{residual}$$

– new trial vectors are generated from the residual until it is sufficiently small

- For fast convergence, we precondition with the simplified response equation

$$\mathbf{E}_{\text{F}}^{(2)}(\tilde{\mathbf{R}}_i) - \omega \mathbf{S}^{(2)}(\tilde{\mathbf{R}}_i) = \mathbf{R}_i$$

– in the MO basis,  $\mathbf{E}_{\text{F}}^{(2)}$  is diagonal (orbital-energy differences) and solution is trivial

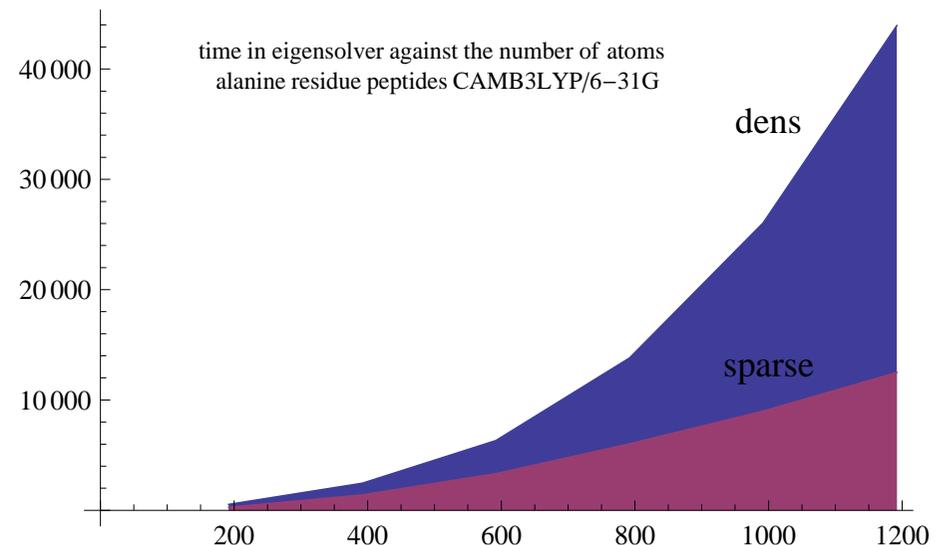
– in the OAO basis,  $\mathbf{E}_{\text{F}}^{(2)}$  is nondiagonal and 5–20 iterations are required for solution

## Excitation energies

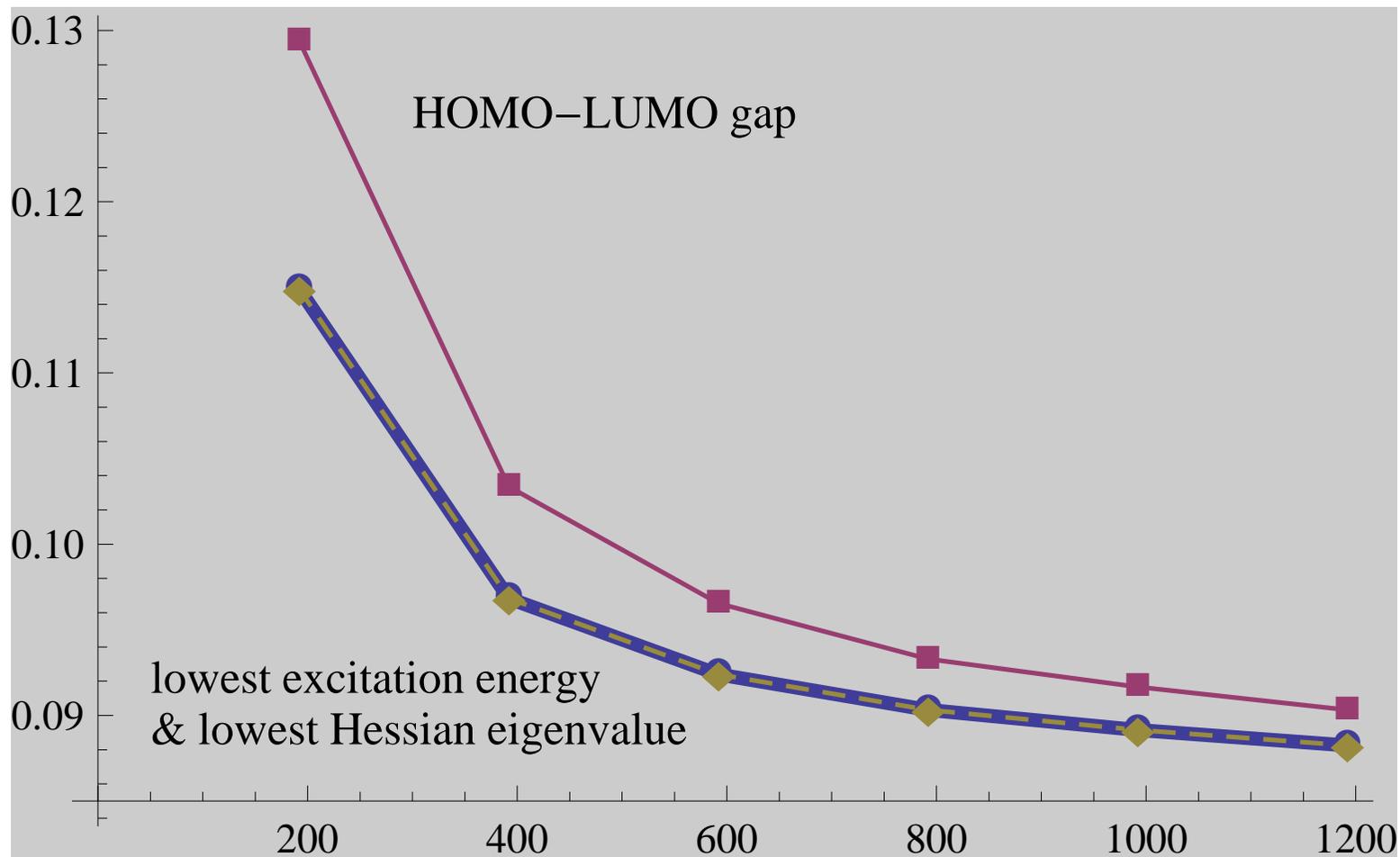
- The calculation of excitation energies (RPA) more difficult than that of polarizabilities
  - the eigenmatrices of the simplified Hessian provide good starting guesses:

$$\mathbf{E}_F^{(2)}(\mathbf{X}_{ia}) = (\epsilon_a - \epsilon_i)\mathbf{S}^{(2)}(\mathbf{X}_{ia}) \quad \text{where} \quad \mathbf{X}_{ia} = \mathbf{C}_i \mathbf{C}_a^T, \quad \begin{cases} \mathbf{F}^{oo} \mathbf{C}_i = \epsilon_i \mathbf{C}_i \\ \mathbf{F}^{vv} \mathbf{C}_a = \epsilon_a \mathbf{C}_a \end{cases}$$

- slow convergence of preconditioning equations (subspace problem nearly singular)
- CAMB3LYP/6-31G alanine residues
- cubic complexity with dense-matrix algebra
- linear complexity with sparse-matrix algebra
- preconditioning part dominates
- Fock/KS matrix construction dominates (not shown here)



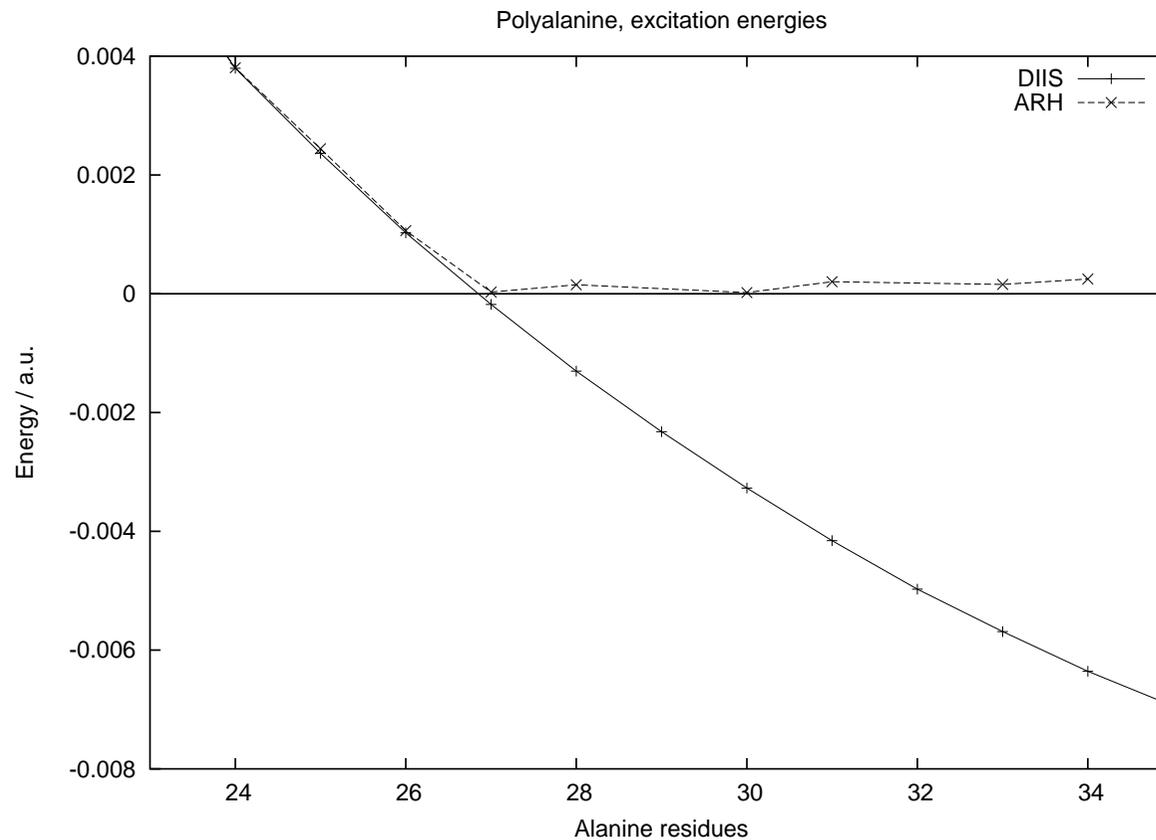
CAMB3LYP/6-31G excitation energies of alanine residue peptides



- lowest excitation energy almost identical with lowest Hessian eigenvalue
- CAMB3LYP has 19% short-range and 65% long-range exact exchange

## B3LYP/6-31G excitation energies of alanine residue peptides

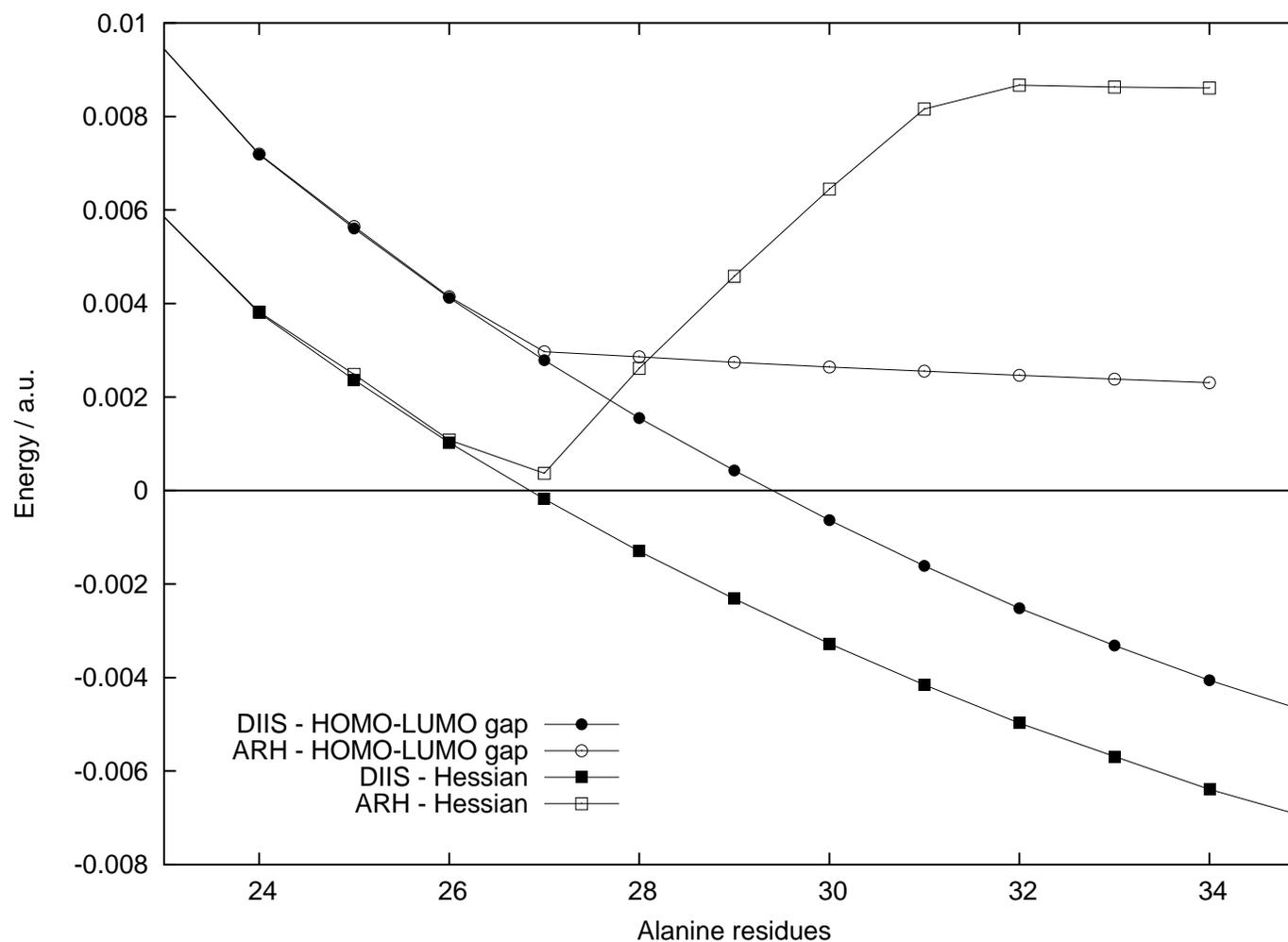
- B3LYP (with 20% exact exchange) behaves differently, in an unphysical manner
  - beyond 27 alanine residues, the lowest excitation energy becomes very small
  - DIIS with a small level shift converges to an excited state (first-order saddle point)



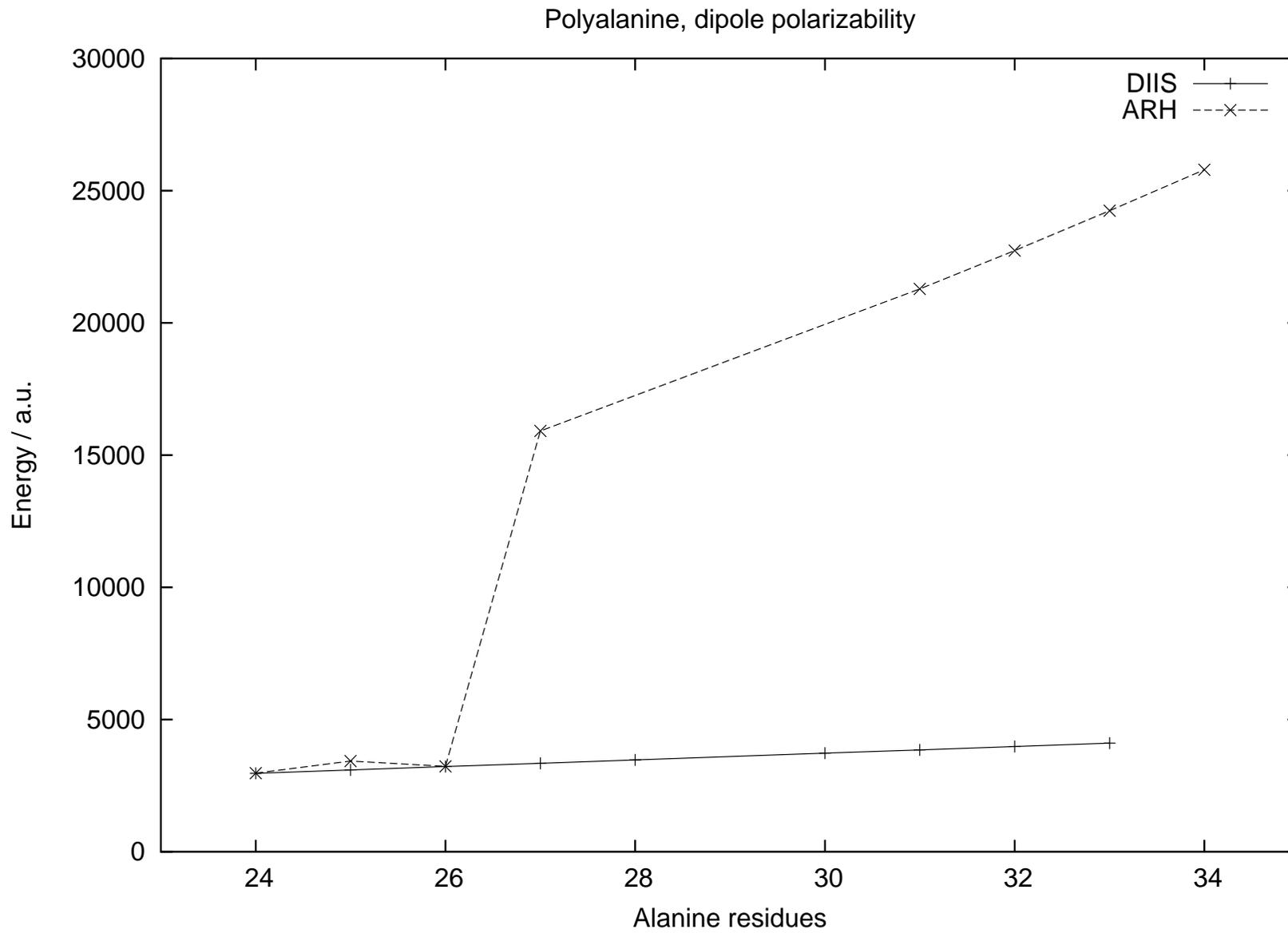
- for LDA and GGA functionals, the crossing occurs even earlier
  - local DFT functionals unable to account for long-range exchange in large molecules

## B3LYP/6-31G polyaniline HOMO–LUMO gap and lowest Hessian eigenvalue

- The B3LYP energy minimum behaves in a strange manner
  - the lowest Hessian eigenvalue goes through a minimum for 27 residues
  - the level-shifted DIIS solution behaves smoothly with increasing chain length

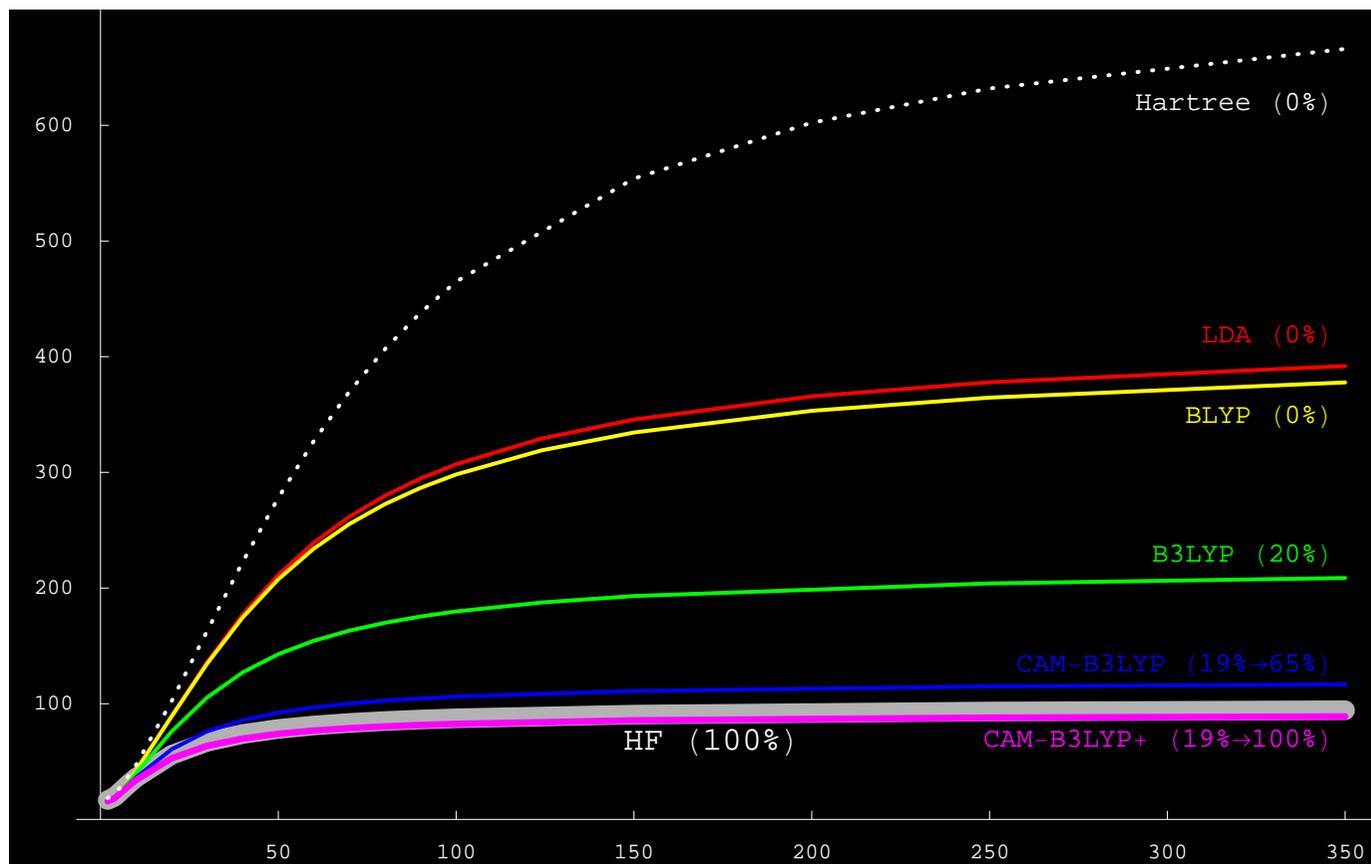


B3LYP/6-31G polyalanine polarizabilities



## Static polarizabilities

- To illustrate, we have calculated longitudinal polarizabilities of linear alkene chains
  - HF and DFT  $\alpha/N$  in 6-31G basis, plotted against the number of carbons  $N$



- poor long-range exchange leads to overpolarization

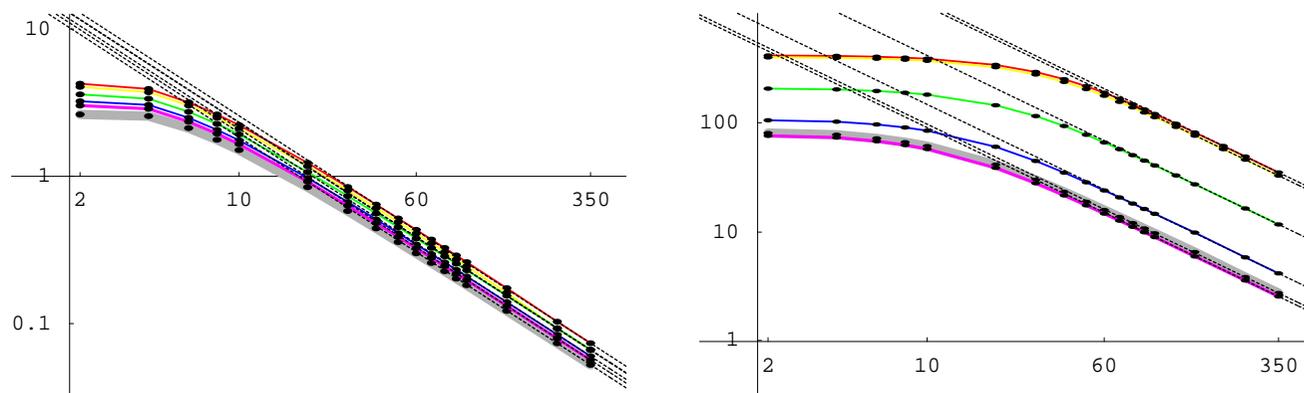
## Asymptotic behaviour of group polarizabilities

- How does the group polarizability converge towards the infinite limit?

$$\bar{\alpha}_{\infty} - \bar{\alpha}_N = eN^{-1} + \mathcal{O}(N^{-2}) \quad \text{Kudin *et al.*, JCP **122**, 134907 (2005)}$$

– this behaviour is universal, holding at all levels of theory

- Log–log plots of  $\bar{\alpha}_{\infty} - \bar{\alpha}_N$  for alkanes and alkenes:



– limit obtained by extrapolation  $\bar{\alpha}_{\infty} = (\alpha_N - \alpha_M)/(N - M)$

– straight lines of slope  $-1$  superimposed through the points at  $N = 350$

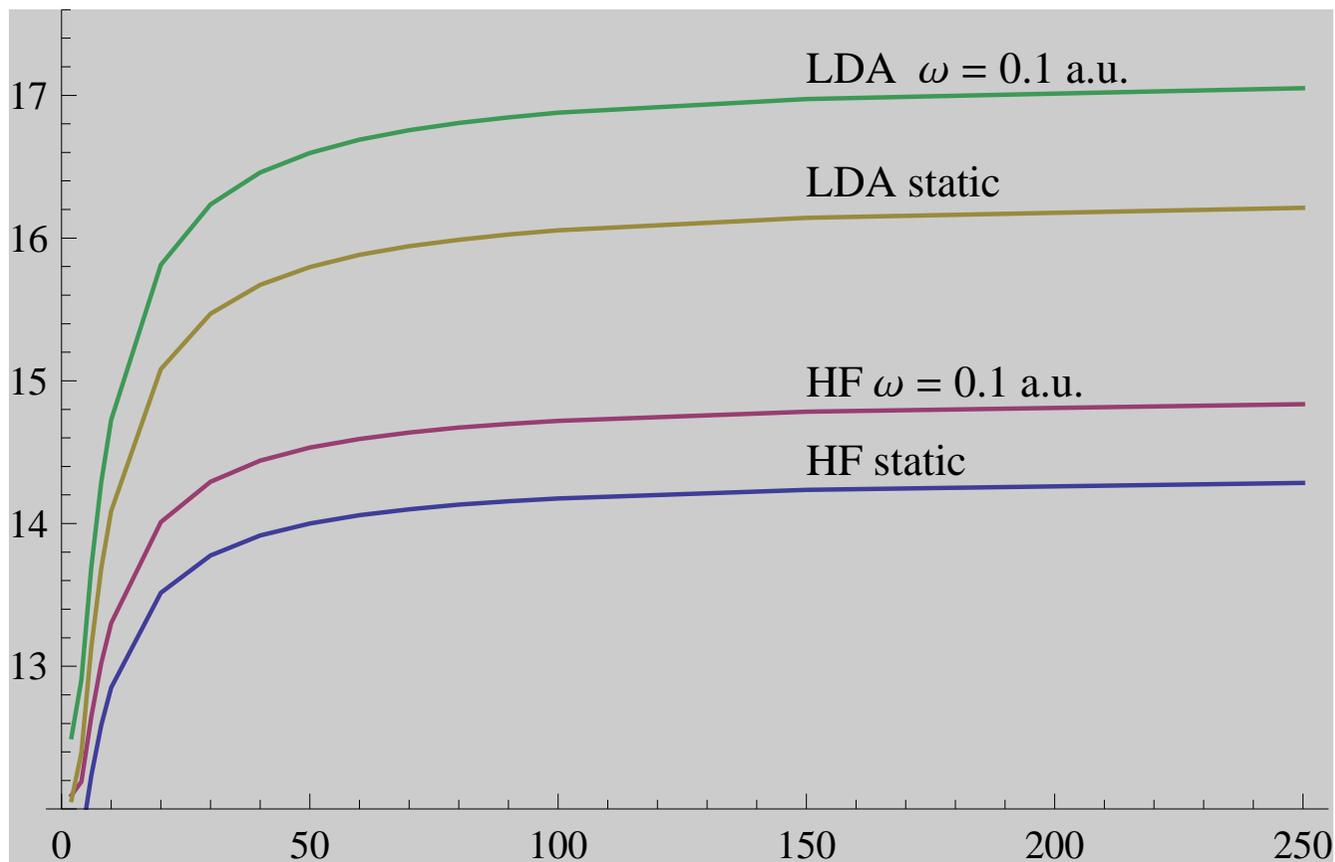
- The asymptotic region is reached with  $\text{C}_{30}\text{H}_{62}$  (alkanes) and  $\text{C}_{60}\text{H}_{62}$  (alkenes)

– alkane  $\bar{\alpha}_{\infty}$  predicted to within 1% from  $\text{C}_{30}\text{H}_{62}$

– alkene  $\bar{\alpha}_{\infty}$  predicted to within 1% from  $\text{C}_{60}\text{H}_{62}$  for HF and from  $\text{C}_{150}\text{H}_{152}$  for LDA

## Frequency-dependent polarizabilities

- HF and LDA longitudinal polarizabilities in linear alkane chains
  - $\alpha/N$  in 6-31G basis, plotted against the number of carbons  $N$



- long-range exchange is less important for the saturated alkanes
- LDA overestimates dispersion as well as the static limit

## Summary

- We have considered several computational tasks in self-consistent field theory
  - orthogonalization of the atomic orbitals
  - solution of the Roothaan–Hall eigenvalue problem
  - linear response theory
- For sufficiently sparse AO matrices, all these tasks can be carried out in linear time
  - the Fock/KS matrix is reevaluated no more often than in MO theory
  - the remaining work is rich in matrix multiplication
  - well suited to parallelization
- We have not considered the global convergence of the SCF method
  - standard methods may not converge to the ground state
  - better global optimization schemes must be developed
  - standard functionals often perform poorly for large systems
- Standard functionals may not be suitable for large systems
  - local functionals do not recover long-range correlation and exchange effects
  - the inclusion of exact exchange important