

Molecular Magnetic Properties

Workshop on Theoretical Chemistry

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Overview

- The electronic Hamiltonian in an electromagnetic field
- The calculation of molecular magnetic properties

Nonrelativistic electronic Hamiltonian in an electromagnetic field

- classical mechanics of particles
 - Newtonian mechanics (1687)
 - Lagrangian mechanics (1788)
 - Hamiltonian mechanics (1833)
- electromagnetic fields
 - Maxwell's equations (1864)
 - scalar and vector potentials
 - gauge transformations
- spin-free electron in an electromagnetic field
 - the Lagrangian and the Hamiltonian of a particle in an electromagnetic field
 - the Schrödinger Hamiltonian (1925)
- spinning electron in an electromagnetic field
 - the Dirac equation and electron spin (1928)
 - the Schrödinger–Pauli Hamiltonian (Pauli, 1927)

Classical mechanics

- Matter is described by **Newton's equations**:

$$\mathbf{F}(\mathbf{r}, \mathbf{v}, t) = m\mathbf{a}$$

- the force defines the system and is obtained from experiment
- conservative forces (e.g., gravitational forces) are obtained from potentials:

$$\mathbf{F}(\mathbf{r}) = -\nabla V(\mathbf{r})$$

- Radiation is described by **Maxwell's equations**:

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0 \quad \leftarrow \text{Coulomb's law}$$

$$\nabla \times \mathbf{B} - \epsilon_0\mu_0 \partial\mathbf{E}/\partial t = \mu_0\mathbf{J} \quad \leftarrow \text{Ampère's law}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} + \partial\mathbf{B}/\partial t = \mathbf{0} \quad \leftarrow \text{Faraday's law of induction}$$

- The interaction between matter and radiation is described by the **Lorentz force**:

$$\mathbf{F}(\mathbf{r}, \mathbf{v}) = z(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Lagrangian mechanics: the principle of least action

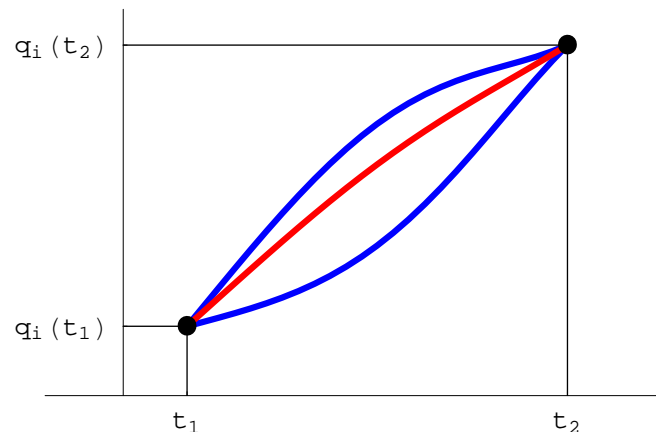
- For a system of n degrees of freedom, there are
 - n **generalized coordinates** q_i in configuration space
 - n **generalized velocities** \dot{q}_i

- The **principle of least action** (Hamilton's principle):

For each system, there exists a Lagrangian $L(q_i, \dot{q}_i, t)$ such that the action integral

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t) dt$$

assumes an extremum along the trajectory in configuration space taken by the system.



Lagrange's equations

- From the principle of least action, we obtain

$$\begin{aligned}\delta S &= \delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t) dt = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) dt \\ &= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_i} \delta q_i - \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \right] dt = 0\end{aligned}$$

- We conclude that the Lagrangian satisfies the following second-order differential equations (one for each degree of freedom):

$$\boxed{\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}} \quad \leftarrow \text{Lagrange's equations of motion}$$

- the Lagrangian defines the system and is determined so that it reproduces the equations of motion consistent with experiment.
- Lagrange's equations preserve their form in any coordinate system.
- A more general formulation than the Newtonian one:
 - unified description of matter and fields (Newton's and Maxwell's equations)
 - springboard for quantum mechanics

Arbitrariness of the Lagrangian and gauge transformations

- The scalar Lagrangian is not uniquely defined.
- Assume that the Lagrangian $L(q, \dot{q}, t)$ satisfies the equations:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}.$$

- Consider now the following transformed Lagrangian where the arbitrary **gauge function** $f(q, t)$ is independent of the velocity \dot{q} :

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{d}{dt} f(q, t).$$

- The new Lagrangian satisfies the **same equations of motion** as the old one:

$$\begin{aligned} \frac{d}{dt} \frac{\partial L'}{\partial \dot{q}} &= \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} + \frac{d}{dt} \frac{\partial}{\partial \dot{q}} \left(\frac{\partial f}{\partial q} \dot{q} + \frac{\partial f}{\partial t} \right) \\ &= \frac{\partial L}{\partial q} + \frac{d}{dt} \frac{\partial f}{\partial q} = \frac{\partial}{\partial q} \left(L + \frac{d}{dt} f \right) = \frac{\partial L'}{\partial q}. \end{aligned}$$

- This is an example of a **gauge transformation**.

Conservative systems

- The Lagrangian of a particle in a **conservative field** may be written as

$$L = \underbrace{T(q, \dot{q})}_{\text{kinetic energy}} - \underbrace{V(q)}_{\text{potential energy}}$$

- The Lagrangian is thus easily set up for any conservative system, in any convenient coordinate system.
- Example: Assuming a Cartesian coordinate system

$$L = \frac{1}{2}mv^2 - V(\mathbf{r}),$$

we find that Lagrange's equations immediately reduce to Newton's equations:

$$\frac{d}{dt} \frac{\partial L}{\partial \mathbf{v}} = \frac{\partial L}{\partial \mathbf{r}} \quad \Rightarrow \quad \frac{d}{dt} m\mathbf{v} = -\nabla V(\mathbf{r}) \quad \Rightarrow \quad m\mathbf{a} = \mathbf{F}.$$

- For particles in a (nonconservative) electromagnetic field, the Lagrangian can be cast in similar but slightly different form as discussed later.

The energy function

- In Lagrangian mechanics, the **energy function** is defined as:

$$h(q, \dot{q}, t) = \frac{\partial L}{\partial \dot{q}} \dot{q} - L(q, \dot{q}, t).$$

- Assume a conservative Cartesian system with Lagrangian $L = \frac{1}{2}mv^2 - V(\mathbf{r})$:

$$h = \frac{\partial L}{\partial \mathbf{v}} \mathbf{v} - L = \frac{\partial T}{\partial \mathbf{v}} \mathbf{v} - (T - V) = 2T - T + V = T + V = \text{total energy}$$

- More generally, h is equal to the **total energy** if $T(\dot{q})$ is quadratic in \dot{q} and $V(q)$ is independent of \dot{q} .

- The energy function h is conserved if L does not depend explicitly on time:

$$\frac{dh}{dt} = -\frac{\partial L}{\partial t}$$

- Proof:

$$\begin{aligned} \frac{dh}{dt} &= \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \dot{q} \right) - \frac{dL}{dt} \\ &= \left[\left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \dot{q} + \frac{\partial L}{\partial \dot{q}} \ddot{q} \right] - \left[\frac{\partial L}{\partial q} \dot{q} + \frac{\partial L}{\partial \dot{q}} \ddot{q} + \frac{\partial L}{\partial t} \right] = -\frac{\partial L}{\partial t} \end{aligned}$$

Generalized momentum

- The **generalized momentum** p_i conjugate to the generalized coordinate q_i is defined as

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad \leftarrow \text{generalized momentum}$$

- For a conservative system in Cartesian coordinates $L = \frac{1}{2}mv^2 - V(\mathbf{r})$, the conjugate momentum corresponds to the **linear momentum**:

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = \frac{\partial T}{\partial \mathbf{v}} = \frac{1}{2}m \frac{\partial v^2}{\partial \mathbf{v}} = m\mathbf{v} \quad \leftarrow \text{linear momentum}$$

- The momentum conjugate to a coordinate that does not occur in the Lagrangian is conserved:

$$\frac{dp_i}{dt} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} = 0 \quad \leftarrow \text{if } L \text{ is independent of } q_i$$

– such coordinates are said to be **cyclic**

- Compare: h is conserved if L does not depend explicitly on t ,
 p_i is conserved if L does not depend explicitly on q_i

Hamiltonian mechanics

- For a system of n degrees of freedom, there are n second-order differential equations (Lagrange's equations):

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$

- The motion is completely specified by the initial values of the n coordinates and the n velocities.
- In this sense, we may view q_i and \dot{q}_i as $2n$ independent variables.
- Alternatively, let us treat q_i and p_i as $2n$ independent variables:

$$\{q_i, \dot{q}_i\} \rightarrow \{q_i, p_i\}$$

- Possible advantages of such a scheme:
 - first-order equations
 - better suited to cyclic coordinates

The Hamiltonian

- The differential of the Lagrangian $L(q, \dot{q}, t)$ is given by

$$dL(q, \dot{q}, t) = \frac{\partial L}{\partial q} dq + \frac{\partial L}{\partial \dot{q}} d\dot{q} + \frac{\partial L}{\partial t} dt = \dot{p} dq + p d\dot{q} + \frac{\partial L}{\partial t} dt$$

- We now introduce the **Hamiltonian**, whose differential should be given by:

$$dH(q, p, t) = \frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial p} dp + \frac{\partial H}{\partial t} dt. \quad (1)$$

- The Legendre transformation

$$H = p\dot{q} - L \quad \leftarrow \text{the Hamiltonian function}$$

gives the required differential:

$$dH = (p d\dot{q} + \dot{q} dp) - (\dot{p} dq + p d\dot{q} + \frac{\partial L}{\partial t} dt) = -\dot{p} dq + \dot{q} dp - \frac{\partial L}{\partial t} dt. \quad (2)$$

- A comparison of (1) and (2) yields

$$\frac{\partial H}{\partial q} = -\dot{p}, \quad \frac{\partial H}{\partial p} = \dot{q} \quad \leftarrow \text{Hamilton's equations}$$

Prescription for setting up the Hamiltonian

1. Choose n generalized coordinates q_i .
2. Set up the Lagrangian $L(q_i, \dot{q}_i, t)$ such that

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} \quad (1)$$

reproduces the equations of motion.

3. Introduce the conjugate momenta

$$p_i = \frac{\partial L}{\partial \dot{q}_i}. \quad (2)$$

4. Construct the Hamiltonian

$$H = \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i, t) \quad (3)$$

and invert (2) to express the Hamiltonian $H(q_i, p_i)$ in terms of q_i and p_i .

5. Write down Hamilton's equations of motion:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}. \quad (4)$$

Comparison of Lagrangian and Hamiltonian mechanics

Lagrangian mechanics	Hamiltonian mechanics
<p>n second-order equations:</p> $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$	<p>$2n$ first-order equations:</p> $\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$
<p>q_i are the primary variables in configuration space, \dot{q}_i secondary variables</p>	<p>q_i and p_i are independent variables in phase space, connected only by the equations of motion</p>
<p>the state of the system is determined when the variables (q_i, \dot{q}_i) are known at a given time t</p>	<p>the state of the system is defined by a point (q_i, p_i) in phase space, moving on a trajectory that satisfies Hamilton's equations of motion</p>

Poisson brackets

- The **Poisson bracket** of two dynamical variables $A(q, p, t)$ and $B(q, p, t)$ is defined as

$$[A, B] = \sum_i \left(\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial B}{\partial q_i} \frac{\partial A}{\partial p_i} \right)$$

- the fundamental Poisson brackets among conjugate coordinates and momenta:

$$[q_i, q_j] = 0, \quad [p_i, p_j] = 0, \quad [q_i, p_j] = \delta_{ij}$$

- The total time derivatives are given by

$$\frac{dA}{dt} = \frac{\partial A}{\partial q} \dot{q} + \frac{\partial A}{\partial p} \dot{p} + \frac{\partial A}{\partial t} = \frac{\partial A}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial A}{\partial p} \frac{\partial H}{\partial q} + \frac{\partial A}{\partial t},$$

and may therefore be expressed compactly as

$$\frac{dA}{dt} = [A, H] + \frac{\partial A}{\partial t}.$$

- important special cases:

$$\dot{q} = [q, H], \quad \dot{p} = [p, H], \quad \frac{dH}{dt} = \frac{\partial H}{\partial t},$$

Quantization of a particle in conservative force field

- The Hamiltonian formulation is more general than the Newtonian formulation:
 - it is invariant to coordinate transformations
 - it provides a uniform description of matter and radiation
 - it constitutes the springboard to quantum mechanics
- The Hamiltonian function (the total energy) of a particle in a conservative force field:

$$H(q, p) = \frac{p^2}{2m} + V(q)$$

- Standard rule for quantization (in Cartesian coordinates):
 - carry out the substitutions

$$\mathbf{p} \rightarrow -i\hbar\nabla, \quad H \rightarrow i\hbar\frac{\partial}{\partial t}$$

- multiply the resulting expression by the wave function $\Psi(q)$ from the right:

$$i\hbar\frac{\partial\Psi(q)}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(q) \right] \Psi(q)$$

- This approach is sufficient for a treatment of electrons in an electrostatic field.
- It is insufficient for nonconservative systems—that is, for systems in a general electromagnetic field.

Review: Hamiltonian mechanics

- In classical Hamiltonian mechanics, a system of particles is described in terms their positions q_i and conjugate momenta p_i .
- For each such system, there exists a scalar Hamiltonian function $H(q_i, p_i)$ such that the classical equations of motion are given by:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad (\text{Hamilton's equations of motion})$$

- Example: a single particle of mass m in a conservative force field $F(q)$
 - the Hamiltonian function is constructed from a scalar potential:

$$H(q, p) = \frac{p^2}{2m} + V(q), \quad F(q) = -\frac{\partial V(q)}{\partial q}$$

- Hamilton's equations are equivalent to Newton's equations:

$$\left. \begin{aligned} \dot{q} &= \frac{\partial H(q, p)}{\partial p} = \frac{p}{m} \\ \dot{p} &= -\frac{\partial H(q, p)}{\partial q} = -\frac{\partial V(q)}{\partial q} \end{aligned} \right\} \Rightarrow m\ddot{q} = F(q) \quad (\text{Newton's equations of motion})$$

- Whereas Newton's equations of motion are second-order differential equations, Hamilton's equations are first-order.
- We must now generalize this approach to particles in an electromagnetic field!

The Lorentz force and Maxwell's equations

- In the presence of an electric field \mathbf{E} and a magnetic field (magnetic induction) \mathbf{B} , a classical particle of charge z experiences the Lorentz force:

$$\mathbf{F} = z(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

– since this force depends on the velocity \mathbf{v} of the particle, it is not conservative

- The electric and magnetic fields \mathbf{E} and \mathbf{B} satisfy Maxwell's equations (1861–1868):

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0 \quad \leftarrow \text{Coulomb's law}$$

$$\nabla \times \mathbf{B} - \epsilon_0\mu_0 \partial\mathbf{E}/\partial t = \mu_0\mathbf{J} \quad \leftarrow \text{Ampère's law with Maxwell's correction}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} + \partial\mathbf{B}/\partial t = \mathbf{0} \quad \leftarrow \text{Faraday's law of induction}$$

– when the charge and current densities $\rho(\mathbf{r}, t)$ and $\mathbf{J}(\mathbf{r}, t)$ are known, Maxwell's equations can be solved for $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$

– on the other hand, since the charges (particles) are driven by the Lorentz force, $\rho(\mathbf{r}, t)$ and $\mathbf{J}(\mathbf{r}, t)$ are functions of $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$

- In the following, we shall consider the motion of particles in a given (fixed) electromagnetic field.

Scalar and vector potentials

- The second, homogeneous pair of Maxwell's equations involve only \mathbf{E} and \mathbf{B} :

$$\nabla \cdot \mathbf{B} = 0 \quad (1)$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0} \quad (2)$$

1. Equation (1) is satisfied by introducing the vector potential \mathbf{A} :

$$\nabla \cdot \mathbf{B} = 0 \Rightarrow \mathbf{B} = \nabla \times \mathbf{A} \quad \leftarrow \text{vector potential} \quad (3)$$

2. Inserting Eq. (3) in Eq. (2) and introducing a scalar potential ϕ , we obtain

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = \mathbf{0} \Rightarrow \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \phi \quad \leftarrow \text{scalar potential}$$

- The second pair of Maxwell's equations are thus automatically satisfied by writing

$$\begin{aligned} \mathbf{E} &= -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} &= \nabla \times \mathbf{A} \end{aligned}$$

- The potentials (ϕ, \mathbf{A}) contain four rather than six components as in (\mathbf{E}, \mathbf{B}) .
- They are obtained by solving the first, inhomogeneous pair of Maxwell's equations, which contains ρ and \mathbf{J} .

Particle in an electromagnetic field

- For a particle in an electromagnetic field, we must set up a Lagrangian such that

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$

reduces to Newton's equations with the Lorentz force

$$\mathbf{F} = z(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \quad \mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}$$

- This is not a conservative system, for which

$$L = T - V, \quad F_i = -\frac{\partial V}{\partial q_i}$$

- Rather, it belongs to a broader class of systems, for which

$$L = T - U, \quad F_i = -\frac{\partial U}{\partial q_i} + \frac{d}{dt} \left(\frac{\partial U}{\partial \dot{q}_i} \right)$$

- For a particle subject to the Lorentz force, the **generalized potential** is given by

$$U = z(\phi - \mathbf{v} \cdot \mathbf{A}) \quad \leftarrow \text{velocity-dependent potential}$$

and the Lagrangian becomes

$$L = T - z(\phi - \mathbf{v} \cdot \mathbf{A}) \quad \leftarrow \text{particle in an electromagnetic field}$$

Conjugate momentum in an electromagnetic field

- We recall that, for a conservative system described by Lagrangian of the form

$$L(q, \dot{q}) = T(\dot{q}) - V(q),$$

the conjugate momentum in Cartesian coordinates

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = \frac{\partial T}{\partial \mathbf{v}} = m\mathbf{v} \equiv \boldsymbol{\pi}$$

is equal to the linear (kinetic) momentum $\boldsymbol{\pi}$:

$$\boxed{\mathbf{p} = \boldsymbol{\pi}} \quad \leftarrow \text{particle in a conservative field}$$

- By contrast, for a nonconservative system described by Lagrangian

$$L(q, \dot{q}) = T(\dot{q}) - U(q, \dot{q}),$$

the conjugate and kinetic momenta are no longer the same.

- In particular, for a particle in an electromagnetic field

$$L = T - z(\phi - \mathbf{v} \cdot \mathbf{A})$$

we obtain

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = \frac{\partial T}{\partial \mathbf{v}} + z\mathbf{A} \quad \Rightarrow \quad \boxed{\mathbf{p} = \boldsymbol{\pi} + z\mathbf{A}} \quad \leftarrow \text{particle in an electromagnetic field}$$

The Hamiltonian in an electromagnetic field

- We recall that, for a conservative system with Lagrangian

$$L = T(\dot{q}) - V(q),$$

where $T(\dot{q})$ is quadratic in \dot{q} , the Hamiltonian is given by

$$H = T(\dot{q}) + V(q).$$

- Let us now consider the nonconservative system consisting of a particle in a field

$$L = T(\dot{q}) - U(q, \dot{q}) = \frac{1}{2}mv^2 - z(\phi - \mathbf{v} \cdot \mathbf{A}).$$

- From the conjugate momentum

$$\mathbf{p} = m\mathbf{v} + z\mathbf{A},$$

we obtain the Hamiltonian

$$H = \mathbf{p} \cdot \mathbf{v} - L = (m\mathbf{v} + z\mathbf{A}) \cdot \mathbf{v} - \left(\frac{1}{2}mv^2 - z\phi + z\mathbf{v} \cdot \mathbf{A} \right) = \frac{1}{2}mv^2 + z\phi = T + z\phi.$$

- Expressed in canonical coordinates, the Hamiltonian now becomes:

$$H = T + z\phi = \frac{(\mathbf{p} - z\mathbf{A}) \cdot (\mathbf{p} - z\mathbf{A})}{2m} + z\phi$$

– note: $H = T + U + z\mathbf{v} \cdot \mathbf{A} \neq T + U$

Gauge transformations

- The scalar and vector potentials ϕ and \mathbf{A} are not unique.
- Consider the following transformation of the potentials:

$$\left. \begin{aligned} \phi' &= \phi - \frac{\partial f}{\partial t} \\ \mathbf{A}' &= \mathbf{A} + \nabla f \end{aligned} \right\} f = f(q, t) \quad \leftarrow \text{gauge function of position and time}$$

- This **gauge transformation** of the potentials does not affect the physical fields:

$$\begin{aligned} \mathbf{E}' &= -\nabla\phi' - \frac{\partial\mathbf{A}'}{\partial t} = -\nabla\phi + \nabla\frac{\partial f}{\partial t} - \frac{\partial\mathbf{A}}{\partial t} - \frac{\partial\nabla f}{\partial t} = \mathbf{E} \\ \mathbf{B}' &= \nabla \times (\mathbf{A} + \nabla f) = \mathbf{B} + \nabla \times \nabla f = \mathbf{B} \end{aligned}$$

- We are free to choose $f(q, t)$ so as to make ϕ and \mathbf{A} satisfy additional conditions.
- In the **Coulomb gauge**, the gauge function is chosen such that the vector potential becomes divergenceless:

$$\nabla \cdot \mathbf{A} = 0 \quad \leftarrow \text{Coulomb gauge}$$

- Note: Gauge transformations induce the following transformations:

$$L' = L + z \frac{df}{dt}, \quad \mathbf{p}' = \mathbf{p} + z \nabla f, \quad H' = H - z \frac{\partial f}{\partial t}, \quad T' = T$$

– However, the equations of motion are unaffected!

Quantization of a particle in an electromagnetic field

- We have now constructed a Hamiltonian function such that Hamilton's equations are equivalent to Newton's equation with the Lorentz force:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \& \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \Leftrightarrow \quad m\mathbf{a} = z(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

- To this end, we introduced scalar and vector potentials ϕ and \mathbf{A} such that

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}$$

- In terms of these potentials, the classical Hamiltonian function takes the form

$$H = \frac{\pi^2}{2m} + z\phi, \quad \boldsymbol{\pi} = \mathbf{p} - z\mathbf{A} \quad \leftarrow \text{kinetic momentum}$$

- Quantization is now accomplished in the usual manner, by the substitutions

$$\mathbf{p} \rightarrow -i\hbar\nabla, \quad H \rightarrow i\hbar\frac{\partial}{\partial t}$$

- This results in the following time-dependent Schrödinger equation for a particle in an electromagnetic field:

$$i\hbar\frac{\partial\Psi}{\partial t} = \frac{1}{2m}(-i\hbar\nabla - z\mathbf{A}) \cdot (-i\hbar\nabla - z\mathbf{A})\Psi + z\phi\Psi$$

Electron spin

- According to our previous discussion, the nonrelativistic Hamiltonian for an electron in an electromagnetic field is given by:

$$H = \frac{\pi^2}{2m} - e\phi, \quad \boldsymbol{\pi} = -i\hbar\nabla + e\mathbf{A}$$

- However, this description ignores a fundamental property of the electron: spin.
- Spin was introduced by Pauli in 1927, to fit experimental observations:

$$H = \frac{(\boldsymbol{\sigma} \cdot \boldsymbol{\pi})^2}{2m} - e\phi = \frac{\pi^2}{2m} + \frac{e\hbar}{2m} \mathbf{B} \cdot \boldsymbol{\sigma} - e\phi$$

where $\boldsymbol{\sigma}$ contains three operators, represented by the two-by-two Pauli spin matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- The Schrödinger equation now becomes a two-component equation

$$\begin{pmatrix} \frac{\pi^2}{2m} - e\phi + \frac{e\hbar}{2m} B_z & \frac{e\hbar}{2m} (B_x - iB_y) \\ \frac{e\hbar}{2m} (B_x + iB_y) & \frac{\pi^2}{2m} - e\phi - \frac{e\hbar}{2m} B_z \end{pmatrix} \begin{pmatrix} \Psi_\alpha \\ \Psi_\beta \end{pmatrix} = E \begin{pmatrix} \Psi_\alpha \\ \Psi_\beta \end{pmatrix}$$

– the two components are only coupled in the presence of an external magnetic field

Spin and relativity

- The introduction of spin by Pauli in 1927 may appear somewhat ad hoc.
- By contrast, spin arises naturally from Dirac's relativistic treatment in 1928.
 - is spin is a relativistic effect?
- However, reduction of Dirac's equation to nonrelativistic form yields the Hamiltonian

$$H = \frac{(\boldsymbol{\sigma} \cdot \boldsymbol{\pi})^2}{2m} - e\phi \neq \frac{\pi^2}{2m} - e\phi$$

- spin is therefore not a relativistic property of the electron
- Indeed, it is possible to take the factorized form

$$H = \frac{(\boldsymbol{\sigma} \cdot \boldsymbol{\pi})^2}{2m} - e\phi$$

as the starting point for a nonrelativistic treatment, with unspecified operators $\boldsymbol{\sigma}$.

- All algebraic properties of $\boldsymbol{\sigma}$ then follow from the requirement $(\boldsymbol{\sigma} \cdot \mathbf{p})^2 = p^2$:

$$[\sigma_i, \sigma_j]_+ = \delta_{ij}, \quad [\sigma_i, \sigma_j] = 2\epsilon_{ijk}\sigma_k$$

- these operators are represented by the two-by-two Pauli spin matrices
- We interpret $\boldsymbol{\sigma}$ by associating an intrinsic angular momentum (spin) with the electron:

$$\mathbf{s} = \hbar\boldsymbol{\sigma}/2$$

Classical relativistic Hamiltonian

- Hamiltonian for an electron in an electromagnetic field

$$H = \sqrt{m^2c^4 + c^2(\mathbf{p} + e\mathbf{A})^2} - e\phi$$

- Hamilton's equations give us

$$\begin{aligned}\dot{\mathbf{r}} &= \frac{\partial H}{\partial \mathbf{p}} \Rightarrow \mathbf{p} = \boldsymbol{\pi} - e\mathbf{A} \quad \leftarrow \text{conjugate momentum} \\ \dot{\mathbf{p}} &= -\frac{\partial H}{\partial \mathbf{r}} \Rightarrow \dot{\boldsymbol{\pi}} = -e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad \leftarrow \text{Lorentz force}\end{aligned}$$

where the relativistic kinetic momentum is given by

$$\boldsymbol{\pi} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}} \quad \leftarrow \text{Lorentz factor} \times \text{nonrelativistic momentum}$$

- Relationship to nonrelativistic mechanics

$$\begin{aligned}\sqrt{m^2c^4 + c^2\boldsymbol{\pi}^2} &= mc^2 + \frac{\boldsymbol{\pi}^2}{2m} + \mathcal{O}[(v/c)^2] \\ \boldsymbol{\pi} &= m\mathbf{v} + \mathcal{O}[(v/c)^2]\end{aligned}$$

– the nonrelativistic limit is obtained as $(v/c)^2 \rightarrow 0$

Linearization of Hamiltonian

- The Hamiltonian is given by

$$H = c\sqrt{\pi^2 + m^2c^2} - e\phi$$

but we would like time and space coordinates to appear symmetrically in the equation

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi$$

- Following Dirac, we write

$$\pi^2 + m^2c^2 = (\alpha_x\pi_x + \alpha_y\pi_y + \alpha_z\pi_z + \alpha_0mc)^2$$

- To determine the α_i , we note that

$$(\alpha_x\pi_x + \alpha_y\pi_y + \dots)^2 = \alpha_x^2\pi_x^2 + \alpha_y^2\pi_y^2 + (\alpha_x\alpha_y + \alpha_y\alpha_x)\pi_x\pi_y + \dots = \pi_x^2 + \pi_y^2 \dots$$

if the α_i operators anticommute

$$\left. \begin{array}{l} \alpha_x^2 = \alpha_y^2 = 1 \\ \alpha_x\alpha_y + \alpha_y\alpha_x = 0 \end{array} \right\} \Rightarrow [\alpha_i, \alpha_j]_+ = 2\delta_{ij}$$

- The Hamiltonian may now be written as

$$H_D = c\boldsymbol{\alpha} \cdot \boldsymbol{\pi} + \beta mc^2 - e\phi, \quad \boldsymbol{\alpha} = [\alpha_x, \alpha_y, \alpha_z], \quad \beta = \alpha_0$$

The Dirac equation

- In matrix representation, the anticommuting operators α_i are represented by four 4×4 matrices

$$\alpha_i = \begin{pmatrix} \mathbf{0} & \sigma_i \\ \sigma_i & \mathbf{0} \end{pmatrix}, \quad \beta = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{pmatrix},$$

where \mathbf{I} is the 2×2 unit matrix and the σ_i are the usual Pauli spin matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- In this representation, the Dirac equation

$$i\hbar \frac{\partial \Psi}{\partial t} = (c\boldsymbol{\alpha} \cdot \boldsymbol{\pi} + \beta mc^2 - e\phi) \Psi$$

therefore has a four-component solution:

$$i\hbar \begin{pmatrix} \frac{\partial \Psi_1}{\partial t} \\ \frac{\partial \Psi_2}{\partial t} \\ \frac{\partial \Psi_3}{\partial t} \\ \frac{\partial \Psi_4}{\partial t} \end{pmatrix} = \begin{pmatrix} mc^2 - e\phi & 0 & c\pi_z & c(\pi_x - i\pi_y) \\ 0 & mc^2 - e\phi & c(\pi_x + i\pi_y) & -c\pi_z \\ c\pi_z & c(\pi_x - i\pi_y) & -mc^2 - e\phi & 0 \\ c(\pi_x + i\pi_y) & -c\pi_z & 0 & -mc^2 - e\phi \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix}$$

- Positive solutions are associated with electrons (α and β spin), negative with positrons.

The Lévy-Leblond equation

- The time-independent Dirac equation may be written in the form:

$$\begin{pmatrix} -e\phi & c\boldsymbol{\sigma} \cdot \boldsymbol{\pi} \\ c\boldsymbol{\sigma} \cdot \boldsymbol{\pi} & -2mc^2 - e\phi \end{pmatrix} \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = (E - mc^2) \begin{pmatrix} \varphi \\ \chi \end{pmatrix}.$$

- Introducing the scaled energy and scaled small component

$$E' = E - mc^2, \quad \chi' = c\chi,$$

and rearranging, we obtain an equation where c occurs only as c^{-2} :

$$\begin{pmatrix} -e\phi & \boldsymbol{\sigma} \cdot \boldsymbol{\pi} \\ \boldsymbol{\sigma} \cdot \boldsymbol{\pi} & -2m - c^{-2}e\phi \end{pmatrix} \begin{pmatrix} \varphi \\ \chi' \end{pmatrix} = E' \begin{pmatrix} \varphi \\ c^{-2}\chi' \end{pmatrix}.$$

- Letting $c \rightarrow \infty$, we obtain the **Lévy-Leblond equation**:

$$\begin{pmatrix} -e\phi & \boldsymbol{\sigma} \cdot \boldsymbol{\pi} \\ \boldsymbol{\sigma} \cdot \boldsymbol{\pi} & -2m \end{pmatrix} \begin{pmatrix} \varphi \\ \chi' \end{pmatrix} = E' \begin{pmatrix} \varphi \\ 0 \end{pmatrix}.$$

- This is the nonrelativistic limit of the Dirac equation—a useful zero-order equation for relativistic perturbation theory.

The Schrödinger equation

- The Lévy-Leblond equation is given by (dropping primes):

$$\begin{pmatrix} -e\phi & \boldsymbol{\sigma} \cdot \boldsymbol{\pi} \\ \boldsymbol{\sigma} \cdot \boldsymbol{\pi} & -2m \end{pmatrix} \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = E \begin{pmatrix} \varphi \\ 0 \end{pmatrix}.$$

- Solving the second equation for the small component χ

$$\chi = \frac{1}{2m} \boldsymbol{\sigma} \cdot \boldsymbol{\pi} \varphi$$

and substituting the result into the first equation, we obtain

$$\left[\frac{1}{2m} (\boldsymbol{\sigma} \cdot \boldsymbol{\pi}) (\boldsymbol{\sigma} \cdot \boldsymbol{\pi}) - e\phi \right] \varphi = E\varphi$$

- Finally, invoking the identity

$$(\boldsymbol{\sigma} \cdot \mathbf{u})(\boldsymbol{\sigma} \cdot \mathbf{v}) = \mathbf{u} \cdot \mathbf{v} + i\boldsymbol{\sigma} \cdot \mathbf{u} \times \mathbf{v}$$

we arrive at the **two-component Schrödinger equation**:

$$\left[\frac{1}{2m} \pi^2 + \frac{1}{2m} i\boldsymbol{\sigma} \cdot \boldsymbol{\pi} \times \boldsymbol{\pi} - e\phi \right] \varphi = E\varphi$$

- In the absence of a vector potential, the second term vanishes:

$$\mathbf{A} = \mathbf{0} \quad \Rightarrow \quad \boldsymbol{\pi} \times \boldsymbol{\pi} = \mathbf{p} \times \mathbf{p} = \mathbf{0}$$

Expansion of the kinetic momentum

- Assuming the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$, we obtain

$$\begin{aligned}
 \pi^2 \Psi &= (\mathbf{p} + e\mathbf{A}) \cdot (\mathbf{p} + e\mathbf{A}) \Psi \\
 &= p^2 \Psi + e\mathbf{p} \cdot \mathbf{A} \Psi + e\mathbf{A} \cdot \mathbf{p} \Psi + e^2 A^2 \Psi \\
 &= p^2 \Psi + e(\mathbf{p} \cdot \mathbf{A}) \Psi + 2e\mathbf{A} \cdot \mathbf{p} \Psi + e^2 A^2 \Psi \\
 &= (p^2 + 2e\mathbf{A} \cdot \mathbf{p} + e^2 A^2) \Psi
 \end{aligned}$$

- Recalling the relation $\nabla \times \mathbf{A} = \mathbf{B}$, we obtain

$$\begin{aligned}
 (\boldsymbol{\pi} \times \boldsymbol{\pi}) \Psi &= (\mathbf{p} + e\mathbf{A}) \times (\mathbf{p} + e\mathbf{A}) \Psi \\
 &= e\mathbf{p} \times \mathbf{A} \Psi + e\mathbf{A} \times \mathbf{p} \Psi \\
 &= e(\mathbf{p} \times \mathbf{A}) \Psi + e(\mathbf{p} \Psi) \times \mathbf{A} + e\mathbf{A} \times \mathbf{p} \Psi \\
 &= -i\hbar e (\nabla \times \mathbf{A}) \Psi = -i\hbar e \mathbf{B} \Psi
 \end{aligned}$$

- In the Coulomb gauge, the kinetic energy operator is therefore given by:

$$T = \frac{1}{2m} \pi^2 + \frac{1}{2m} i\boldsymbol{\sigma} \cdot \boldsymbol{\pi} \times \boldsymbol{\pi} = \frac{1}{2m} p^2 + \frac{e}{m} \mathbf{A} \cdot \mathbf{p} + \frac{e}{m} \mathbf{B} \cdot \mathbf{s} + \frac{e^2}{2m} A^2$$

where we have used $\hbar\boldsymbol{\sigma} = 2\mathbf{s}$.

Electron spin

- The Dirac Hamiltonian does not commute with the orbital angular momentum operator but rather with the operator

$$1 + \frac{\hbar}{2}\boldsymbol{\sigma}$$

- We therefore assign to the electron an **intrinsic spin angular momentum**

$$\mathbf{s} = \frac{\hbar}{2}\boldsymbol{\sigma}$$

- Likewise, we interpret the Zeeman term by assigning to the electron a magnetic moment

$$\mu_B \boldsymbol{\sigma} \cdot \mathbf{B} = -\mathbf{m} \cdot \mathbf{B}, \quad \mu_B = \frac{e\hbar}{2m}$$

where we have introduced is the **anomalous spin magnetic moment**:

$$\mathbf{m} = -2\mu_B \mathbf{s}$$

- From quantum electrodynamics, one finds that the true spin magnetic moment differs slightly from that given by Dirac's theory:

$$\mathbf{m} = -g\mu_B \mathbf{s}, \quad g \approx 2.002$$

The nonrelativistic electronic Hamiltonian

- The general form of the nonrelativistic electronic Hamiltonian is

$$H = \frac{\pi^2}{2m} + \frac{e\hbar}{2m} \mathbf{B} \cdot \boldsymbol{\sigma} - e\phi, \quad \boldsymbol{\pi} = -i\hbar \boldsymbol{\nabla} + e\mathbf{A}$$

- Assuming the Coulomb gauge $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$, we expand the squared kinetic momentum:

$$H = \frac{p^2}{2m} + \frac{e}{m} (\mathbf{A} \cdot \mathbf{p} + \mathbf{B} \cdot \mathbf{s}) + \frac{e^2}{2m} A^2 - e\phi, \quad \mathbf{s} = \frac{\hbar}{2} \boldsymbol{\sigma} \quad \leftarrow \text{spin operator}$$

- For a many-electron system, we add the instantaneous Coulomb interactions and obtain:

$$H_{\text{mol}} = H_0 - \sum_i \phi_i + \sum_i \mathbf{A}_i \cdot \mathbf{p}_i + \sum_i \mathbf{B}_i \cdot \mathbf{s}_i + \frac{1}{2} \sum_i A_i^2$$

where (in atomic units) H_0 is the spin- and field-free molecular electronic Hamiltonian and where the perturbation operators are given by:

1. $-\phi_i$ real singlet
 2. $\mathbf{A}_i \cdot \mathbf{p}_i$ paramagnetic imaginary singlet
 3. $\mathbf{B}_i \cdot \mathbf{s}_i$ paramagnetic real triplet
 4. $\frac{1}{2} A_i^2$ diamagnetic real singlet
- We shall in the following consider nuclear magnetic fields and uniform external magnetic fields.

Uniform static electric and magnetic fields

- The scalar and vector potentials of the uniform (static) fields \mathbf{E} and \mathbf{B} are given by:

$$\left. \begin{aligned} \mathbf{E} &= -\nabla\phi(\mathbf{r}, t) - \frac{\partial\mathbf{A}(\mathbf{r}, t)}{\partial t} = \text{const} \\ \mathbf{B} &= \nabla \times \mathbf{A}(\mathbf{r}, t) = \text{const} \end{aligned} \right\} \Rightarrow \begin{cases} \phi(\mathbf{r}) = -\mathbf{E} \cdot \mathbf{r} \\ \mathbf{A}(\mathbf{r}) = \frac{1}{2}\mathbf{B} \times \mathbf{r} \end{cases}$$

- Interaction with the electrostatic field:

$$-\sum_i \phi(\mathbf{r}_i) = \mathbf{E} \cdot \sum_i \mathbf{r}_i = -\mathbf{E} \cdot \mathbf{d}_e, \quad \mathbf{d}_e = -\sum_i \mathbf{r}_i \quad \leftarrow \text{electric dipole operator}$$

- Orbital paramagnetic interaction with the magnetostatic field:

$$\sum_i \mathbf{A} \cdot \mathbf{p}_i = \frac{1}{2} \sum_i \mathbf{B} \times \mathbf{r}_i \cdot \mathbf{p}_i = \frac{1}{2}\mathbf{B} \cdot \mathbf{L}, \quad \mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i \quad \leftarrow \text{orbital ang. mom. op.}$$

- Spin paramagnetic interaction with the magnetostatic field:

$$\sum_i \mathbf{B} \cdot \mathbf{s}_i = \mathbf{B} \cdot \mathbf{S}, \quad \mathbf{S} = \sum_i \mathbf{s}_i \quad \leftarrow \text{spin ang. mom. op.}$$

- Total paramagnetic interaction with a uniform magnetic field:

$$H_Z = -\mathbf{B} \cdot \mathbf{d}_m, \quad \mathbf{d}_m = -\frac{1}{2}(\mathbf{L} + 2\mathbf{S}) \quad \leftarrow \text{Zeeman interaction}$$

Nuclear magnetic fields and hyperfine interactions

- The nuclear moments set up a magnetic vector potential ($\approx 10^{-8}$ a.u.):

$$\mathbf{A}(\mathbf{r}) = \alpha^2 \sum_K \frac{\mathbf{M}_K \times \mathbf{r}_K}{r_K^3}, \quad \alpha^2 = c^{-2} \approx 10^{-4} \text{ a.u.}, \quad \mathbf{M}_K = \gamma_K \hbar \mathbf{I}_K \approx 10^{-4} \text{ a.u.}$$

- This vector potential gives rise to the following paramagnetic hyperfine interaction

$$\mathbf{A} \cdot \mathbf{p} = \sum_K \mathbf{M}_K^T \mathbf{h}_K^{\text{PSO}}, \quad \mathbf{h}_K^{\text{PSO}} = \alpha^2 \frac{\mathbf{r}_K \times \mathbf{p}}{r_K^3} = \alpha^2 \frac{\mathbf{L}_K}{r_K^3} \quad \leftarrow \text{paramagnetic SO (PSO)}$$

- Taking the curl of this vector potential, we obtain:

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) = \frac{8\pi\alpha^2}{3} \sum_K \delta(\mathbf{r}_K) \mathbf{M}_K + \alpha^2 \sum_K \frac{3\mathbf{r}_K(\mathbf{r}_K \cdot \mathbf{M}_K) - r_K^2 \mathbf{M}_K}{r_K^5}$$

- the first term contributes only when the electron is at the position of the nuclei
- the second term is a classical dipole field and contributes at a distance

- This magnetic field $\mathbf{B}(\mathbf{r})$ then gives rise to two distinct **first-order triplet operators**:

$$\mathbf{B} \cdot \mathbf{s} = \sum_K \mathbf{M}_K^T (\mathbf{h}_K^{\text{FC}} + \mathbf{h}_K^{\text{SD}}), \quad \left\{ \begin{array}{ll} \mathbf{h}_K^{\text{FC}} = \frac{8\pi\alpha^2}{3} \delta(\mathbf{r}_K) \mathbf{s} & \text{Fermi contact (FC)} \\ \mathbf{h}_K^{\text{SD}} = \alpha^2 \frac{3\mathbf{r}_K \mathbf{r}_K^T - r_K^2 \mathbf{I}_3}{r_K^5} \mathbf{s} & \text{spin-dipole (SD)} \end{array} \right.$$

Review

- The nonrelativistic electronic Hamiltonian:

$$H = H_0 + H^{(1)} + H^{(2)} = H_0 + \mathbf{A}(\mathbf{r}) \cdot \mathbf{p} + \mathbf{B}(\mathbf{r}) \cdot \mathbf{s} + \frac{1}{2}A(\mathbf{r})^2$$

- Rayleigh–Schrödinger perturbation theory to second order:

$$E^{(1)} = \langle 0 | \mathbf{A} \cdot \mathbf{p} + \mathbf{B} \cdot \mathbf{s} | 0 \rangle$$

$$E^{(2)} = \frac{1}{2} \langle 0 | A^2 | 0 \rangle - \sum_n \frac{\langle 0 | \mathbf{A} \cdot \mathbf{p} + \mathbf{B} \cdot \mathbf{s} | n \rangle \langle n | \mathbf{A} \cdot \mathbf{p} + \mathbf{B} \cdot \mathbf{s} | 0 \rangle}{E_n - E_0}$$

- Vector potentials of the **uniform external field** and the **nuclear magnetic moments**:

$$\mathbf{A}(\mathbf{r}) = \frac{1}{2} \mathbf{B} \times \mathbf{r}_O, \quad \mathbf{A}_K(\mathbf{r}) = \alpha^2 \frac{\mathbf{M}_K \times \mathbf{r}_K}{r_K^3}, \quad \nabla \times \mathbf{A}(\mathbf{r}) = \mathbf{B}(\mathbf{r}), \quad \nabla \cdot \mathbf{A}(\mathbf{r}) = 0$$

- Orbital and spin **Zeeman interactions** with the external magnetic field:

$$H_{\text{Zeeman}}^{(1)} = \frac{1}{2} \mathbf{B} \cdot \mathbf{L}_O + \mathbf{B} \cdot \mathbf{s}$$

- Orbital and spin **hyperfine interactions** with the nuclear magnetic moments:

$$H_{\text{hyperfine}}^{(1)} = \underbrace{\alpha^2 \frac{\mathbf{M}_K \cdot \mathbf{L}_K}{r_K^3}}_{\text{PSO}} + \underbrace{\frac{8\pi\alpha^2}{3} \delta(\mathbf{r}_K) \mathbf{M}_K \cdot \mathbf{s}}_{\text{FC}} + \underbrace{\alpha^2 \frac{3(\mathbf{s} \cdot \mathbf{r}_K)(\mathbf{r}_K \cdot \mathbf{M}_K) - (\mathbf{M}_K \cdot \mathbf{s})r_K^2}{r_K^5}}_{\text{SD}}$$

Gauge transformation of the Schrödinger equation

- Consider a general gauge transformation:

$$\mathbf{A}' = \mathbf{A} + \nabla f, \quad \phi' = \phi - \frac{\partial f}{\partial t}$$

- It can be shown that the Hamiltonian then transforms in the following manner

$$H' = H + \frac{\partial f}{\partial t},$$

which constitutes a unitary transformation:

$$\left(H' - i \frac{\partial}{\partial t} \right) = \exp(-if) \left(H - i \frac{\partial}{\partial t} \right) \exp(if)$$

- In order that the Schrödinger equation is still satisfied

$$\left(H' - i \frac{\partial}{\partial t} \right) \Psi' \Leftrightarrow \left(H - i \frac{\partial}{\partial t} \right) \Psi,$$

the new wave function must be related to the old one by a compensating unitary transformation:

$$\Psi' = \exp(-if) \Psi$$

- No observable properties such as the electron density are then affected:

$$\rho' = (\Psi')^* \Psi' = [\Psi \exp(-if)]^* [\exp(-if) \Psi] = \Psi^* \Psi = \rho$$

Gauge-origin transformations

- Different choices of gauge origin in the external vector potential

$$\mathbf{A}_{\mathbf{O}}(\mathbf{r}) = \frac{1}{2}\mathbf{B} \times (\mathbf{r} - \mathbf{O})$$

are related by a divergenceless gauge transformation:

$$\mathbf{A}_{\mathbf{K}}(\mathbf{r}) = \mathbf{A}_{\mathbf{O}}(\mathbf{r}) - \mathbf{A}_{\mathbf{O}}(\mathbf{K}) = \mathbf{A}_{\mathbf{O}}(\mathbf{r}) + \nabla f, \quad f(\mathbf{r}) = -\mathbf{A}_{\mathbf{O}}(\mathbf{K}) \cdot \mathbf{r}$$

- the exact wave function transforms accordingly and gives gauge-invariant results:

$$\Psi_{\mathbf{K}}^{\text{exact}} = \exp[i\mathbf{A}_{\mathbf{O}}(\mathbf{K}) \cdot \mathbf{r}] \Psi_{\mathbf{O}}^{\text{exact}}$$

- approximate wave functions are in general not able to carry out this transformation:

$$\Psi_{\mathbf{K}}^{\text{approx}} \neq \exp[i\mathbf{A}_{\mathbf{O}}(\mathbf{K}) \cdot \mathbf{r}] \Psi_{\mathbf{O}}^{\text{approx}}$$

- different gauge origins therefore give different results

- We might contemplate attaching an explicit phase factor to the wave function:

$$\Psi_{\mathbf{K}}^{\text{approx}} \stackrel{\text{def}}{=} \exp[i\mathbf{A}_{\mathbf{O}}(\mathbf{K}) \cdot \mathbf{r}] \Psi_{\mathbf{O}}^{\text{approx}}$$

- for any \mathbf{K} , this approach produces the same result as with the gauge origin at \mathbf{O}
- however, no natural, best gauge origin can usually be identified (except for atoms)
- in any case, we might as well carry out the calculation with the origin at \mathbf{O} !
- applied to **individual AOs**, however, this approach makes much more sense!

Natural gauge origin for AOs

- Assume AOs positioned at \mathbf{K} with the following properties:

$$H_0 \chi_{lm} = E_0 \chi_{lm}, \quad L_z^{\mathbf{K}} \chi_{lm} = m_l \chi_{lm}, \quad \mathbf{L}^{\mathbf{K}} = -i(\mathbf{r} - \mathbf{K}) \times \nabla$$

- We first choose the gauge origin to be at \mathbf{K} :

$$\mathbf{A}_{\mathbf{K}}(\mathbf{r}) = \frac{1}{2} \mathbf{B} \times (\mathbf{r} - \mathbf{K})$$

- The AOs χ_{lm} are then correct to first order in \mathbf{B} :

$$H_{\mathbf{K}}(\mathbf{B}) \chi_{lm} = \left[H_0 + \frac{1}{2} B L_z^{\mathbf{K}} + \mathcal{O}(B^2) \right] \chi_{lm} = \left[E_0 + \frac{1}{2} m_l B + \mathcal{O}(B^2) \right] \chi_{lm}$$

- Next, we put the gauge origin at $\mathbf{O} \neq \mathbf{K}$:

$$\mathbf{A}_{\mathbf{O}}(\mathbf{r}) = \frac{1}{2} \mathbf{B} \times (\mathbf{r} - \mathbf{O})$$

- The AOs χ_{lm} are now correct only to zero order in \mathbf{B} :

$$H_{\mathbf{O}}(\mathbf{B}) \chi_{lm} = \left[H_0 + \frac{1}{2} B L_z^{\mathbf{O}} + \mathcal{O}(B^2) \right] \chi_{lm} \neq \left[E_0 + \frac{1}{2} m_l B + \mathcal{O}(B^2) \right] \chi_{lm}$$

- Standard AOs are biased towards \mathbf{K} !

London orbitals

- A traditional AO gives best description with the gauge origin at its position \mathbf{K} .
- Attach to each AO a phase factor that represents the gauge-origin transformation to its position \mathbf{K} from the global origin \mathbf{O} :

$$\omega_{lm} = \exp [i\mathbf{A}_{\mathbf{K}}(\mathbf{O}) \cdot \mathbf{r}] \chi_{lm} = \exp [i\frac{1}{2}\mathbf{B} \times (\mathbf{O} - \mathbf{K}) \cdot \mathbf{r}] \chi_{lm}$$

- Each AO now behaves as if the global gauge origin were at its position!
- In particular, all AOs are now correct to first order in \mathbf{B} , for any global origin \mathbf{O} .
- The calculations become gauge-origin independent and uniform (good) quality is guaranteed.
- These are the London orbitals (1937), also known as GIAOs (gauge-origin independent AOs).

Review

- The nonrelativistic electronic Hamiltonian:

$$H = H_0 + H^{(1)} + H^{(2)} = H_0 + \mathbf{A}(\mathbf{r}) \cdot \mathbf{p} + \mathbf{B}(\mathbf{r}) \cdot \mathbf{s} + \frac{1}{2}A(\mathbf{r})^2$$

- Rayleigh–Schrödinger perturbation theory to second order:

$$E^{(1)} = \langle 0 | \mathbf{A} \cdot \mathbf{p} + \mathbf{B} \cdot \mathbf{s} | 0 \rangle$$

$$E^{(2)} = \frac{1}{2} \langle 0 | A^2 | 0 \rangle - \sum_n \frac{\langle 0 | \mathbf{A} \cdot \mathbf{p} + \mathbf{B} \cdot \mathbf{s} | n \rangle \langle n | \mathbf{A} \cdot \mathbf{p} + \mathbf{B} \cdot \mathbf{s} | 0 \rangle}{E_n - E_0}$$

- Vector potentials of the **uniform external field** and the **nuclear magnetic moments**:

$$\mathbf{A}(\mathbf{r}) = \frac{1}{2} \mathbf{B} \times \mathbf{r}_O, \quad \mathbf{A}_K(\mathbf{r}) = \alpha^2 \frac{\mathbf{M}_K \times \mathbf{r}_K}{r_K^3}, \quad \nabla \times \mathbf{A}(\mathbf{r}) = \mathbf{B}(\mathbf{r}), \quad \nabla \cdot \mathbf{A}(\mathbf{r}) = 0$$

- Orbital and spin **Zeeman interactions** with the external magnetic field:

$$H_{\text{Zeeman}}^{(1)} = \frac{1}{2} \mathbf{B} \cdot \mathbf{L}_O + \mathbf{B} \cdot \mathbf{s}$$

- Orbital and spin **hyperfine interactions** with the nuclear magnetic moments:

$$H_{\text{hyperfine}}^{(1)} = \underbrace{\alpha^2 \frac{\mathbf{M}_K \cdot \mathbf{L}_K}{r_K^3}}_{\text{PSO}} + \underbrace{\frac{8\pi\alpha^2}{3} \delta(\mathbf{r}_K) \mathbf{M}_K \cdot \mathbf{s}}_{\text{FC}} + \underbrace{\alpha^2 \frac{3(\mathbf{s} \cdot \mathbf{r}_K)(\mathbf{r}_K \cdot \mathbf{M}_K) - (\mathbf{M}_K \cdot \mathbf{s})r_K^2}{r_K^5}}_{\text{SD}}$$

Taylor expansion of the energy

- Expand the energy in the presence of an external magnetic field \mathbf{B} and nuclear magnetic moments \mathbf{M}_K around zero field and zero moments:

$$\begin{aligned}
 E(\mathbf{B}, \mathbf{M}) = & E_0 + \overbrace{\mathbf{B}^T \mathbf{E}^{(10)}}^{\text{perm. magnetic moments}} + \overbrace{\sum_K \mathbf{M}_K^T \mathbf{E}_K^{(01)}}^{\text{hyperfine coupling}} \\
 & + \underbrace{\frac{1}{2} \mathbf{B}^T \mathbf{E}^{(20)} \mathbf{B}}_{\text{- magnetizability}} + \underbrace{\frac{1}{2} \sum_K \mathbf{B}^T \mathbf{E}_K^{(11)} \mathbf{M}_K}_{\text{shieldings + 1}} + \underbrace{\frac{1}{2} \sum_{KL} \mathbf{M}_K^T \mathbf{E}_{KL}^{(02)} \mathbf{M}_L}_{\text{spin-spin couplings}} + \dots
 \end{aligned}$$

- The first-order terms vanish for closed-shell systems because of symmetry:

$$\langle \text{c.c.} | \hat{\Omega}_{\text{imaginary}} | \text{c.c.} \rangle \equiv \langle \text{c.c.} | \hat{\Omega}_{\text{triplet}} | \text{c.c.} \rangle \equiv 0$$

- Higher-order terms are negligible since the perturbations are tiny:
 - 1) the magnetic induction \mathbf{B} is weak ($\approx 10^{-5}$ a.u.)
 - 2) the nuclear magnetic moments \mathbf{M}_K are small ($\mu_0 \mu_N \approx 10^{-8}$ a.u.)
- We shall therefore consider only the second-order terms:
the magnetizability, the shieldings, and the spin-spin couplings

The magnetizability

- Assume zero nuclear magnetic moments and expand the molecular electronic energy in the external magnetic induction \mathbf{B} :

$$E(\mathbf{B}) = E_0 + \mathbf{B}^T \mathbf{E}^{(10)} + \frac{1}{2} \mathbf{B}^T \mathbf{E}^{(20)} \mathbf{B} + \dots$$

- The molecular magnetic moment at \mathbf{B} is now given by

$$\mathbf{M}_{\text{mol}}(\mathbf{B}) \stackrel{\text{def}}{=} -\frac{dE(\mathbf{B})}{d\mathbf{B}} = -\mathbf{E}^{(10)} - \mathbf{E}^{(20)} \mathbf{B} + \dots = \mathbf{M}_{\text{perm}} + \boldsymbol{\xi} \mathbf{B} + \dots,$$

where we have introduced the permanent magnetic moment and the magnetizability:

$$\mathbf{M}_{\text{perm}} = -\mathbf{E}^{(10)} = -\left. \frac{dE}{d\mathbf{B}} \right|_{\mathbf{B}=0} \quad \leftarrow \text{permanent magnetic moment}$$

– describes the first-order change in the energy but vanishes for closed-shell systems

$$\boldsymbol{\xi} = -\mathbf{E}^{(20)} = -\left. \frac{d^2 E}{d\mathbf{B}^2} \right|_{\mathbf{B}=0} \quad \leftarrow \text{molecular magnetizability}$$

– describes the second-order energy and the first-order induced magnetic moment

- The magnetizability is responsible for molecular diamagnetism, important for molecules without a permanent magnetic moment.

The calculation of magnetizabilities

- The molecular magnetizability of a closed-shell system:

$$\begin{aligned} \xi &= -\frac{d^2 E}{d\mathbf{B}^2} = -\left\langle 0 \left| \frac{\partial^2 H}{\partial \mathbf{B}^2} \right| 0 \right\rangle + 2 \sum_n \frac{\left\langle 0 \left| \frac{\partial H}{\partial \mathbf{B}} \right| n \right\rangle \left\langle n \left| \frac{\partial H}{\partial \mathbf{B}} \right| 0 \right\rangle}{E_n - E_0} \\ &= \underbrace{\frac{1}{4} \left\langle 0 \left| \mathbf{r}_O \mathbf{r}_O^T - (\mathbf{r}_O^T \mathbf{r}_O) \mathbf{I}_3 \right| 0 \right\rangle}_{\text{diamagnetic term}} + \underbrace{\frac{1}{2} \sum_n \frac{\left\langle 0 \left| \mathbf{L}_O \right| n \right\rangle \left\langle n \left| \mathbf{L}_O^T \right| 0 \right\rangle}{E_n - E_0}}_{\text{paramagnetic term}} \end{aligned}$$

- The (often) dominant diamagnetic term arises from differentiation of the operator:

$$\frac{1}{2} A^2(\mathbf{B}) = \frac{1}{8} (\mathbf{B} \times \mathbf{r}_O) \cdot (\mathbf{B} \times \mathbf{r}_O) = \frac{1}{8} [B^2 r_O^2 - (\mathbf{B} \cdot \mathbf{r}_O)(\mathbf{B} \cdot \mathbf{r}_O)]$$

- the isotropic part of the diamagnetic contribution is given by:

$$\xi_{\text{dia}} = \frac{1}{3} \text{Tr} \boldsymbol{\xi}_{\text{dia}} = -\frac{1}{6} \left\langle 0 \left| x_O^2 + y_O^2 + z_O^2 \right| 0 \right\rangle = -\frac{1}{6} \left\langle 0 \left| r_O^2 \right| 0 \right\rangle \quad \leftarrow \text{system surface}$$

- Only the orbital Zeeman interaction contributions to the paramagnetic term:

$$\mathbf{S} |0\rangle \equiv 0 \quad \leftarrow \text{singlet state}$$

- for 1S systems (closed-shell atoms), the paramagnetic term vanishes altogether:

$$\frac{1}{2} \mathbf{L}_O |^1S\rangle \equiv 0 \quad \leftarrow \text{gauge origin at nucleus}$$

Hartree–Fock magnetizabilities

- Basis-set requirements for magnetizabilities are modest if London orbitals are used:

basis	cc-pVDZ	cc-pVTZ	cc-pVQZ
HF basis-set error	2.8%	1.0%	0.4%

- The HF model overestimates the magnitude of magnetizabilities by 5%–10%:

10^{-30} JT^{-2}	HF	exp.	diff.
H ₂ O	−232	−218	−6.4%
NH ₃	−289	−271	−6.6%
CH ₄	−315	−289	−9.0%
CO ₂	−374	−349	−7.2%
PH ₃	−441	−435	−1.4%
H ₂ S	−446	−423	−5.4%
C ₃ H ₄	−482	−420	−14.8%
CSO	−595	−538	−10.6%
CS ₂	−752	−701	−7.3%

– compare with polarizabilities, which require large basis sets and are underestimated

High-resolution NMR spin Hamiltonian

- Consider a molecule in the presence of an external field B along the z axis and with nuclear spins \mathbf{I}_K related to the nuclear magnetic moments \mathbf{M}_K as:

$$\mathbf{M}_K = \gamma_K \hbar \mathbf{I}_K \approx 10^{-4} \text{ a.u.}$$

where γ_K is the magnetogyric ratio of the nucleus.

- Assuming free molecular rotation, the nuclear magnetic energy levels can be reproduced by the following **high-resolution NMR spin Hamiltonian**:

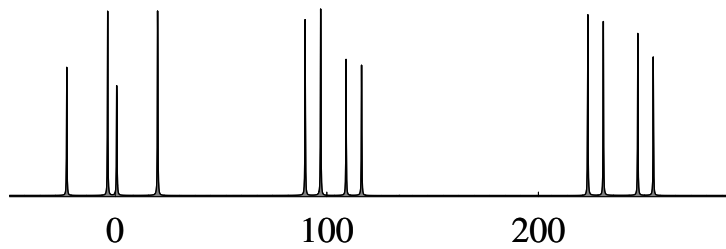
$$H_{\text{NMR}} = \underbrace{- \sum_K \gamma_K \hbar (1 - \sigma_K) B I_{Kz}}_{\text{nuclear Zeeman interaction}} + \underbrace{\sum_{K>L} \gamma_K \gamma_L \hbar^2 K_{KL} \mathbf{I}_K \cdot \mathbf{I}_L}_{\text{nuclear spin-spin interaction}}$$

where we have introduced

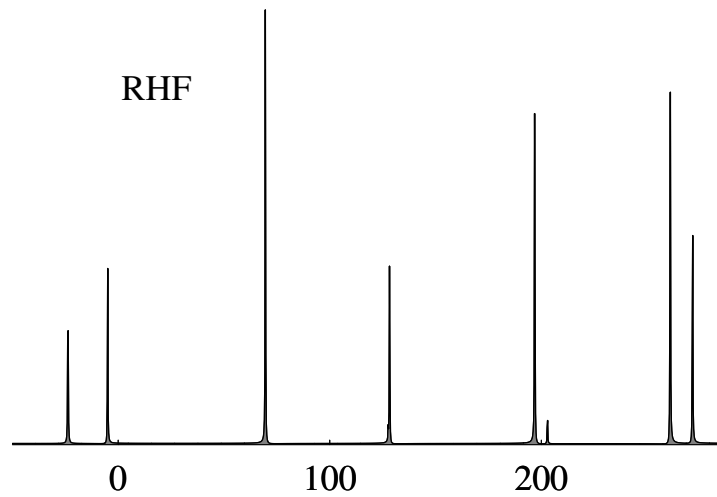
- the **nuclear shielding constants** σ_K
- the **(reduced) indirect nuclear spin-spin coupling constants** K_{KL}
- This is an **effective** nuclear spin Hamiltonian:
 - it reproduces NMR spectra without considering the electrons explicitly
 - the spin parameters σ_K and K_{KL} are adjusted to fit the observed spectra
 - we shall consider their evaluation from molecular electronic-structure theory

Simulated 200 MHz NMR spectra of vinyl lithium

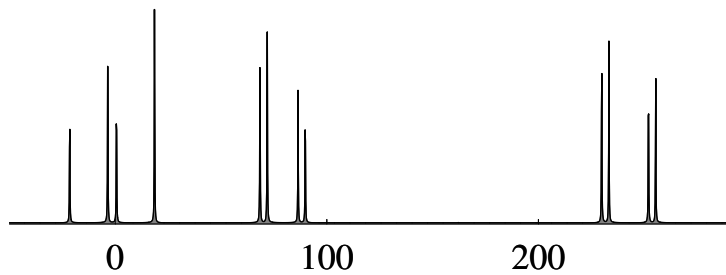
experiment



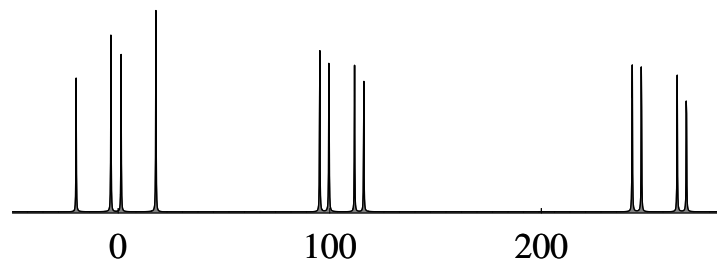
RHF



MCSCF



B3LYP



Nuclear shielding constants

- Recall the energy expansion for a closed-shell molecule in the presence of an external field \mathbf{B} and nuclear magnetic moments \mathbf{M}_K :

$$E(\mathbf{B}, \mathbf{M}) = E_0 + \frac{1}{2} \mathbf{B}^T \mathbf{E}^{(20)} \mathbf{B} + \frac{1}{2} \sum_K \mathbf{B}^T \mathbf{E}_K^{(11)} \mathbf{M}_K + \frac{1}{2} \sum_{KL} \mathbf{M}_K^T \mathbf{E}_{KL}^{(02)} \mathbf{M}_L + \dots$$

- In this expansion, $\mathbf{E}_K^{(11)}$ describes the coupling between the applied field and the nuclear magnetic moments:
 - in the absence of electrons (i.e., in vacuum), this coupling is identical to $-\mathbf{I}_3$:

$$H_{\text{Zeeman}}^{\text{nuc}} = -\mathbf{B} \cdot \sum_K \mathbf{M}_K \quad \leftarrow \text{the purely nuclear Zeeman interaction}$$

- in the presence of electrons (i.e., in a molecule), the coupling is modified slightly:

$$\mathbf{E}_K^{(11)} = -\mathbf{I}_3 + \boldsymbol{\sigma}_K \quad \leftarrow \text{the nuclear shielding tensor}$$

- Since the nuclear shielding constants arise from a hyperfine interaction between the electrons and the nuclei, it is proportional to $\alpha^2 \approx 5 \cdot 10^{-5}$ and is measured in ppm.
- The nuclear Zeeman interaction, which does not enter the electronic problem, has here been introduced in a purely ad hoc fashion. Its status is otherwise similar to that of the Coulomb nuclear–nuclear repulsion operator.

The calculation of nuclear shielding tensors

- Nuclear shielding tensors of a closed-shell system:

$$\begin{aligned}
 \sigma_K &= \frac{d^2 E_{\text{el}}}{d\mathbf{B}d\mathbf{M}_K} = \left\langle 0 \left| \frac{\partial^2 H}{\partial \mathbf{B} \partial \mathbf{M}_K} \right| 0 \right\rangle - 2 \sum_n \frac{\langle 0 | \frac{\partial H}{\partial \mathbf{B}} | n \rangle \langle n | \frac{\partial H}{\partial \mathbf{M}_K} | 0 \rangle}{E_n - E_0} \\
 &= \underbrace{\frac{\alpha^2}{2} \left\langle 0 \left| \frac{\mathbf{r}_O^T \mathbf{r}_K \mathbf{I}_3 - \mathbf{r}_O \mathbf{r}_K^T}{r_K^3} \right| 0 \right\rangle}_{\text{diamagnetic term}} - \underbrace{\alpha^2 \sum_n \frac{\langle 0 | \mathbf{L}_O | n \rangle \langle n | r_K^{-3} \mathbf{L}_K^T | 0 \rangle}{E_n - E_0}}_{\text{paramagnetic term}}
 \end{aligned}$$

- The (usually) dominant diamagnetic term arises from differentiation of the operator:

$$\mathbf{A}(\mathbf{B}) \cdot \mathbf{A}(\mathbf{M}_K) = \frac{1}{2} \alpha^2 r_K^{-3} (\mathbf{B} \times \mathbf{r}_O) \cdot (\mathbf{M}_K \times \mathbf{r}_K)$$

- As for the magnetizability, there is no spin contribution for singlet states:

$$\mathbf{S} |0\rangle \equiv 0 \quad \leftarrow \text{singlet state}$$

- For 1S systems (closed-shell atoms), the paramagnetic term vanishes completely and the the shielding is given by (assuming gauge origin at the nucleus):

$$\sigma_{\text{Lamb}} = \frac{1}{3} \alpha^2 \left\langle ^1S \left| r_K^{-1} \right| ^1S \right\rangle \quad \leftarrow \text{Lamb formula}$$

Benchmark calculations of BH shieldings

	HF	MP2	CCSD	CCSD(T)	FCI
$\sigma(^{11}\text{B})$	-261.3	-220.7	-166.6	-170.5	-170.1
$\sigma(^1\text{H})$	24.21	24.12	24.74	24.62	24.60
$\Delta\sigma(^{11}\text{B})$	690.1	629.9	549.4	555.2	554.7
$\Delta\sigma(^1\text{H})$	14.15	14.24	13.52	13.69	13.70

- TZP+ basis, $R_{\text{BH}} = 123.24$ pm
- J. Gauss and K. Ruud, *Int. J. Quantum Chem.* **S29** (1995) 437
- J. Gauss and J. F. Stanton, *J. Chem. Phys.* **104** (1996) 2574

Calculated and experimental equilibrium shielding constants

		HF	CAS	MP2	CCSD	CCSD(T)	exp.
HF	F	413.6	419.6	424.2	418.1	418.6	410 ± 6
	H	28.4	28.5	28.9	29.1	29.2	28.5 ± 0.2
H ₂ O	O	328.1	335.3	346.1	336.9	337.9	344 ± 17
	H	30.7	30.2	30.7	30.9	30.9	30.05 ± 0.02
NH ₃	N	262.3	269.6	276.5	269.7	270.7	264.5
	H	31.7	31.0	31.4	31.6	31.6	31.2 ± 1.0
CH ₄	C	194.8	200.4	201.0	198.7	198.9	198.7
	H	31.7	31.2	31.4	31.5	31.6	30.61
F ₂	F	-167.9	-136.6	-170.0	-171.1	-186.5	-192.8
N ₂	N	-112.4	-53.0	-41.6	-63.9	-58.1	-61.6 ± 0.2
CO	C	-25.5	8.2	10.6	0.8	5.6	3.0 ± 0.9
	O	-87.7	-38.9	-46.5	-56.0	-52.9	-42.3 ± 17

- For references and details, see Helgaker, Jaszuński, and Ruud, *Chem. Rev.* **99** (1999) 293.

DFT shielding constants

		HF	LDA	BLYP	B3LYP	KT2	CCSD(T)	exp.
HF	F	413.6	416.2	401.0	408.1	411.4	418.6	410 ± 6
H ₂ O	O	328.1	334.8	318.2	325.0	329.5	337.9	344 ± 17
NH ₃	N	262.3	266.3	254.6	259.2	264.6	270.7	264.5
CH ₄	C	194.8	193.1	184.2	188.1	195.1	198.9	198.7
F ₂	F	-167.9	-284.2	-336.7	-208.3	-211.0	-186.5	-192.8
N ₂	N	-112.4	-91.4	-89.8	-86.4	-59.7	-58.1	-61.6 ± 0.2
CO	C	-25.5	-20.3	-19.3	-17.5	7.4	5.6	3.0 ± 0.9
	O	-87.7	-87.5	-85.4	-78.1	-57.1	-52.9	-42.3 ± 17

Coupled-cluster convergence of shielding constants

	CCSD	CCSD(T)	CCSDT	CCSDTQ	CCSDTQ5	FCI
$\sigma(^{13}\text{C})$	32.23	35.91	35.66	36.10	36.14	36.15
$\Delta\sigma(^{13}\text{C})$	361.30	356.10	356.47	355.85	355.80	355.79
$\sigma(^{17}\text{O})$	-13.93	-13.03	-13.16	-12.81	-12.91	-12.91
$\Delta\sigma(^{17}\text{O})$	636.01	634.55	634.75	634.22	634.52	634.35

- All calculations in the cc-pVDZ basis and with a frozen core.
- Kállay and Gauss, *J. Chem. Phys.* **120** (2004) 6841.

Nuclear spin–spin couplings

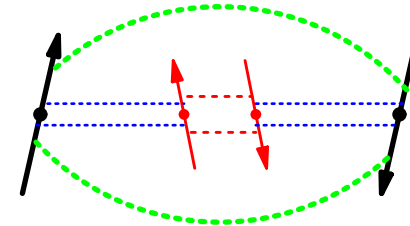
- The last term in the expansion of the molecular electronic energy in \mathbf{B} and \mathbf{M}_K

$$E(\mathbf{B}, \mathbf{M}) = E_0 + \frac{1}{2} \mathbf{B}^T \mathbf{E}^{(20)} \mathbf{B} + \frac{1}{2} \sum_K \mathbf{B}^T \mathbf{E}_K^{(11)} \mathbf{M}_K + \frac{1}{2} \sum_{KL} \mathbf{M}_K^T \mathbf{E}_{KL}^{(02)} \mathbf{M}_L + \dots$$

describes the coupling of the nuclear magnetic moments in the presence of electrons.

- There are two distinct contributions to the coupling:
the **direct and indirect** contributions

$$\mathbf{E}_{KL}^{(02)} = \mathbf{D}_{KL} + \mathbf{K}_{KL}$$



- The **direct coupling** occurs by a classical dipole mechanism:

$$\mathbf{D}_{KL} = \alpha^2 R_{KL}^{-5} (R_{KL}^2 \mathbf{I}_3 - 3 \mathbf{R}_{KL} \mathbf{R}_{KL}^T) \approx 10^{-12} \text{ a.u.}$$

– it is anisotropic and vanishes in isotropic media such as gases and liquids

- The **indirect coupling** arises from **hyperfine interactions** with the surrounding electrons:

– it is exceedingly small: $\mathbf{K}_{KL} \approx 10^{-16} \text{ a.u.} \approx 1 \text{ Hz}$

– it does not vanish in isotropic media

– it gives the fine structure of high-resolution NMR spectra

- Experimentalists usually work in terms of the (nonreduced) spin–spin couplings

$$\mathbf{J}_{KL} = h \frac{\gamma_K}{2\pi} \frac{\gamma_L}{2\pi} \mathbf{K}_{KL} \quad \leftarrow \text{isotope dependent}$$

The calculation of indirect nuclear spin–spin coupling tensors

- The indirect nuclear spin–spin coupling tensor of a closed-shell system is given by:

$$\mathbf{K}_{KL} = \frac{d^2 E_{el}}{d\mathbf{M}_K d\mathbf{M}_L} = \left\langle 0 \left| \frac{\partial^2 H}{\partial \mathbf{M}_K \partial \mathbf{M}_L} \right| 0 \right\rangle - 2 \sum_n \frac{\left\langle 0 \left| \frac{\partial H}{\partial \mathbf{M}_K} \right| n \right\rangle \left\langle n \left| \frac{\partial H}{\partial \mathbf{M}_L} \right| 0 \right\rangle}{E_n - E_0}$$

- Carrying out the differentiation, we obtain:

$$\begin{aligned} \mathbf{K}_{KL} = & \underbrace{\alpha^4 \left\langle 0 \left| \frac{\mathbf{r}_K^T \mathbf{r}_L \mathbf{I}_3 - \mathbf{r}_K \mathbf{r}_L^T}{r_K^3 r_L^3} \right| 0 \right\rangle}_{\text{diamagnetic spin-orbit (DSO)}} - \underbrace{2\alpha^4 \sum_n \frac{\left\langle 0 \left| r_K^{-3} \mathbf{L}_K \right| n \right\rangle \left\langle n \left| r_L^{-3} \mathbf{L}_L^T \right| 0 \right\rangle}{E_n - E_0}}_{\text{paramagnetic spin-orbit (PSO)}} \\ & - \underbrace{2\alpha^4 \sum_n \frac{\left\langle 0 \left| \frac{8\pi}{3} \delta(\mathbf{r}_K) \mathbf{s} + \frac{3\mathbf{r}_K \mathbf{r}_K^T - r_K^2 \mathbf{I}_3}{r_K^5} \mathbf{s} \right| n \right\rangle \left\langle n \left| \frac{8\pi}{3} \delta(\mathbf{r}_L) \mathbf{s}^T + \frac{3\mathbf{r}_L \mathbf{r}_L^T - r_L^2 \mathbf{I}_3}{r_L^5} \mathbf{s}^T \right| 0 \right\rangle}{E_n - E_0}}_{\text{Fermi contact (FC) and spin-dipole (SD)}} \end{aligned}$$

- the isotropic FC/FC term often dominates short-range coupling constants
- the FC/SD and SD/FC terms often dominate the anisotropic part of \mathbf{K}_{KL}
- the orbital contributions (especially DSO) are usually but not invariably small
- for large internuclear separations, the DSO and PSO contributions cancel

Calculations of indirect nuclear spin–spin coupling constants

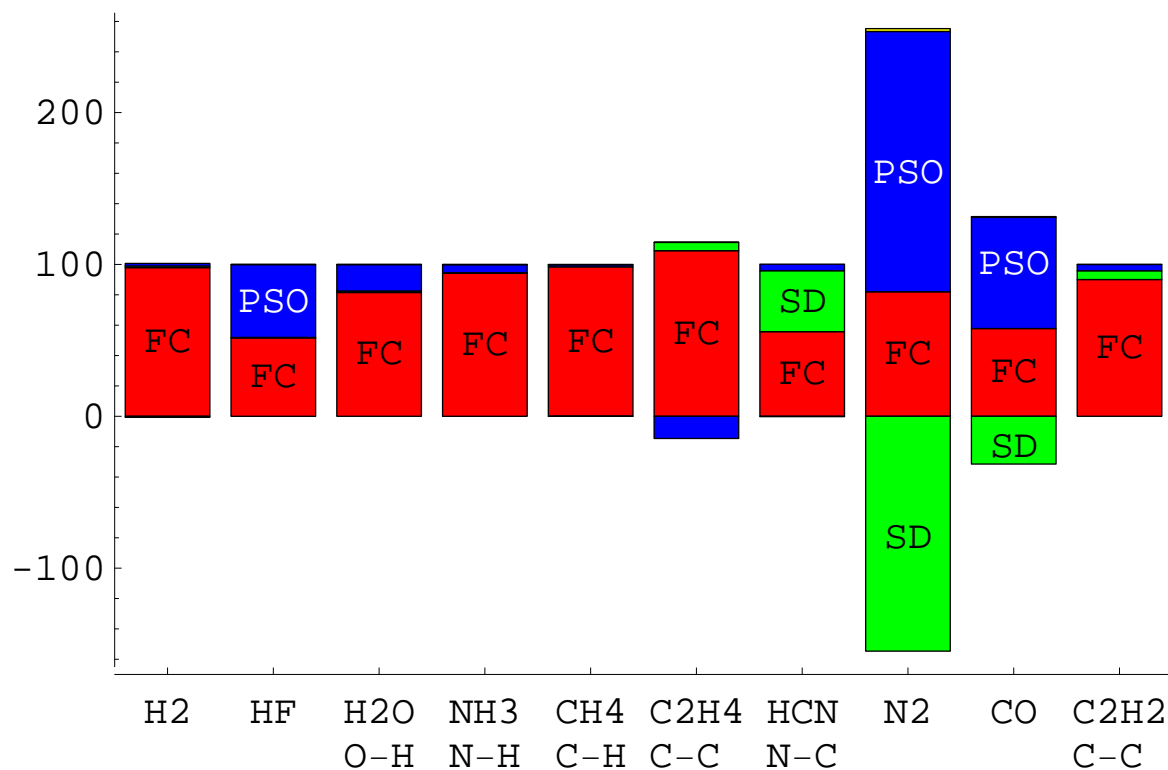
- The calculation of spin–spin coupling constants is a **challenging task**:
 - triplet as well as singlet perturbations are involved
 - electron correlation important—the Hartree–Fock model fails abysmally
 - the dominant FC contribution requires an accurate description of the electron density at the nuclei (large decontracted s sets)
- We must solve a large number of response equations:
 - 3 singlet equations and 7 triplet equations for each nucleus
 - for shieldings, only 3 equations are required, for molecules of all sizes
- Spin–spin couplings are very sensitive to the molecular geometry:
 - equilibrium structures must be chosen carefully
 - large vibrational corrections (often 5%–10%)
- However, unlike in shielding calculations, there is no need for London orbitals since no external magnetic field is involved.
- For heavy elements, a relativistic treatment may be necessary.

Relative importance of the contributions to spin–spin coupling constants

- The isotropic indirect spin–spin coupling constants can be uniquely decomposed as:

$$\mathbf{J}_{KL} = \mathbf{J}_{KL}^{\text{DSO}} + \mathbf{J}_{KL}^{\text{PSO}} + \mathbf{J}_{KL}^{\text{FC}} + \mathbf{J}_{KL}^{\text{SD}}$$

- The spin–spin coupling constants are often dominated by the FC term.
- Since the FC term is relatively easy to calculate, it is tempting to ignore the other terms.
- However, none of the contributions can be *a priori* neglected (N₂ and CO)!

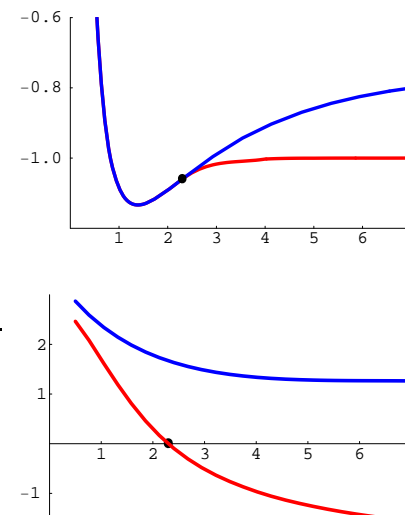


RHF and the triplet instability problem

- RHF **does not** in general work for spin–spin calculations:

- the RHF wave function often becomes triplet unstable
- at or close to such instabilities, the RHF description of spin interactions becomes unphysical
- the spin–spin coupling constants of C₂H₄:

Hz	¹ J _{CC}	¹ J _{CH}	² J _{CH}	² J _{HH}	³ J _{cis}	³ J _{trans}
exp.	68	156	–2	2	12	19
RHF	1270	755	–572	–344	360	400
CAS	76	156	–1	3	14	21



- Indeed, any method **based on the RHF reference state** may have problems:

- ¹J_{CN} in HCN [Auer and Gauss, JCP 115 (2001) 1619]

Hz	RHF	CCSD	CCSD(T)	CC3	CCSDT
relaxed	–92.0	–8.1	7.4	–2.6	–14.5
unrelaxed		–15.0		–14.7	–14.6

- in CC theory, orbital relaxation should be treated through singles amplitudes
- noniterative CCSD(T) should not be used; use iterative CC3 instead
- all electrons should be correlated in unrelaxed CC calculations

Reduced spin–spin coupling constants ($10^{19} \text{kg m}^{-2} \text{s}^{-2} \text{\AA}^{-2}$)

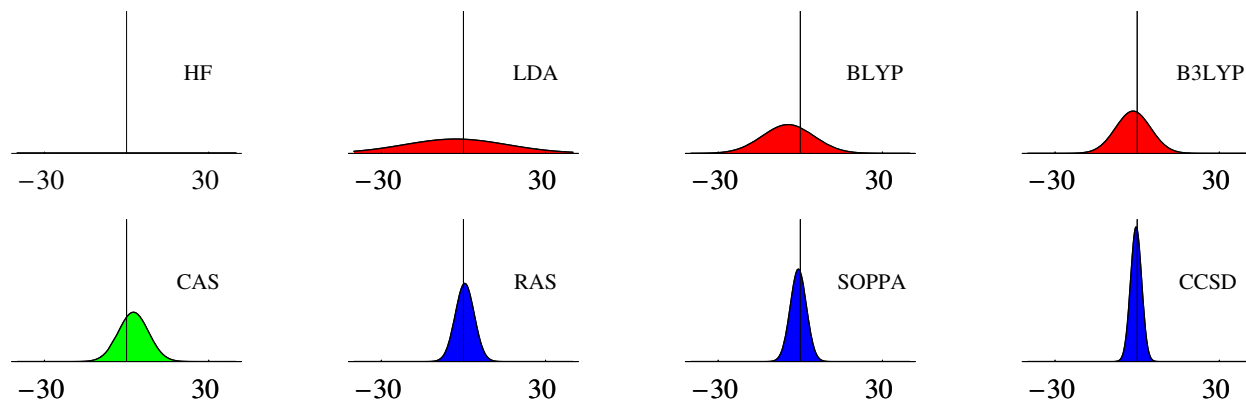
		RHF	CAS	RAS	SOPPA	CCSD	CC3	exp*	vib
HF	$^1J_{\text{HF}}$	59.2	48.0	48.1	46.8	46.1	46.1	47.6	−3.4
CO	$^1J_{\text{CO}}$	13.4	−28.1	−39.3	−45.4	−38.3	−37.3	−38.3	−1.7
N ₂	$^1J_{\text{NN}}$	175.0	−5.7	−9.1	−23.9	−20.4	−20.4	−19.3	−1.1
H ₂ O	$^1J_{\text{OH}}$	63.7	51.5	47.1	49.5	48.4	48.2	52.8	−3.3
	$^2J_{\text{HH}}$	−1.9	−0.8	−0.6	−0.7	−0.6	−0.6	−0.7	0.1
NH ₃	$^1J_{\text{NH}}$	61.4	48.7	50.2	51.0	48.1		50.8	−0.3
	$^2J_{\text{HH}}$	−1.9	−0.8	−0.9	−0.9	−1.0		−0.9	0.1
C ₂ H ₄	$^1J_{\text{CC}}$	1672.0	99.6	90.5	92.5	92.3		87.8	1.2
	$^1J_{\text{CH}}$	249.7	51.5	50.2	52.0	50.7		50.0	1.7
	$^2J_{\text{CH}}$	−189.3	−1.9	−0.5	−1.0	−1.0		−0.4	−0.4
	$^2J_{\text{HH}}$	−28.7	−0.2	0.1	0.1	0.0		0.2	0.0
	$^3J_{\text{cis}}$	30.0	1.0	1.0	1.0	1.0		0.9	0.1
	$^3J_{\text{tns}}$	33.3	1.5	1.5	1.5	1.5		1.4	0.2
$ \bar{\Delta} $	abs.	180.3	3.3	1.6	1.8	1.2	1.6	*at R_e	
	%	5709	60	14	24	23	6		

Reduced spin–spin coupling constants ($10^{19} \text{kg m}^{-2} \text{s}^{-2} \text{\AA}^{-2}$)

		RHF	LDA	BLYP	B3LYP	RAS	exp*	vib
HF	$^1 J_{\text{HF}}$	59.2	35.0	34.5	36.8	48.1	47.6	−3.4
CO	$^1 J_{\text{CO}}$	13.4	−65.4	−55.7	−44.9	−39.3	−38.3	−1.7
N ₂	$^1 J_{\text{NN}}$	175.0	32.9	−46.6	−32.9	−9.1	−19.3	−1.1
H ₂ O	$^1 J_{\text{OH}}$	63.7	40.3	44.6	46.6	47.1	52.8	−3.3
	$^2 J_{\text{HH}}$	−1.9	−0.3	−0.9	−0.6	−0.6	−0.7	0.1
NH ₃	$^1 J_{\text{NH}}$	61.4	41.0	49.6	52.6	50.2	50.8	−0.3
	$^2 J_{\text{HH}}$	−1.9	−0.4	−0.7	−0.8	−0.9	−0.9	0.1
C ₂ H ₄	$^1 J_{\text{CC}}$	1672.0	66.6	90.3	98.3	90.5	87.8	1.2
	$^1 J_{\text{CH}}$	249.7	42.5	55.3	54.7	50.2	50.0	1.7
	$^2 J_{\text{CH}}$	−189.3	0.4	0.0	−0.4	−0.5	−0.4	−0.4
	$^2 J_{\text{HH}}$	−28.7	0.4	0.4	0.2	0.1	0.2	0.0
	$^3 J_{\text{cis}}$	30.0	0.8	1.1	1.1	1.0	0.9	0.1
	$^3 J_{\text{tns}}$	33.3	1.2	1.7	1.7	1.5	1.4	0.2
$ \bar{\Delta} $	abs.	180.3	11.2	5.9	4.2	1.6	*at R_e	
	%	5709	72	48	20	14		

Comparison of density-functional and wave-function theory

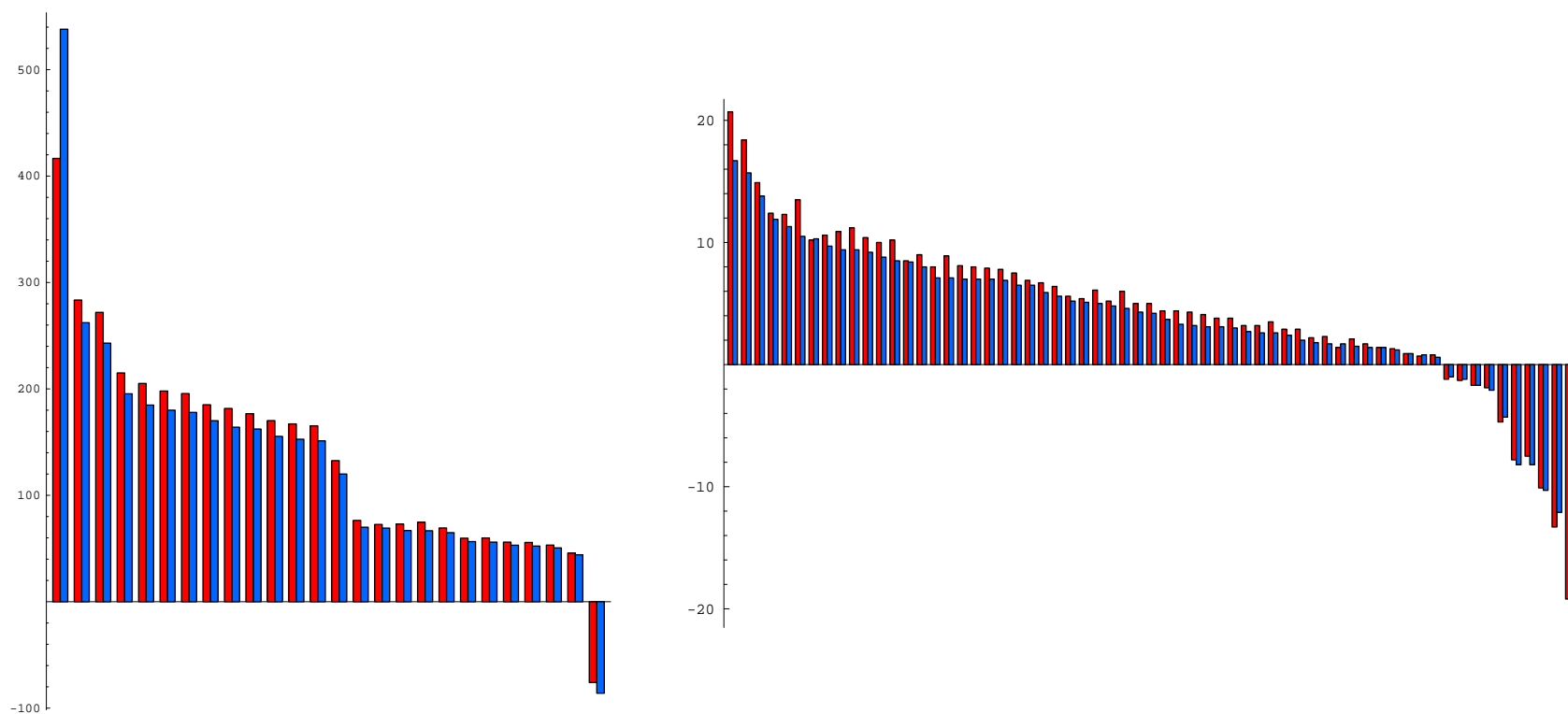
- normal distributions of errors for these molecules and some other systems for which vibrational corrections have been made:



- some observations:
 - HF has a very broad distribution and overestimates strongly
 - LDA underestimates only slightly, but has a large standard deviation
 - BLYP reduces the LDA errors by a factor of two
 - B3LYP improves upon GGA (but not as dramatically as for other properties)
 - B3LYP errors are similar to those of CASSCF and about twice those of the dynamically correlated methods RASSCF, SOPPA, and CCSD
 - the most accurate method appears to be CCSD
 - the situation is much less satisfactory than for geometries and atomization energies

Trends in spin–spin coupling constants

- comparison of calculated (red) and B3LYP (blue) spin–spin coupling constants
 - plotted in order of decreasing experimental value



- trends are quite well reproduced by B3LYP, in particular for large couplings

Basis-set requirements

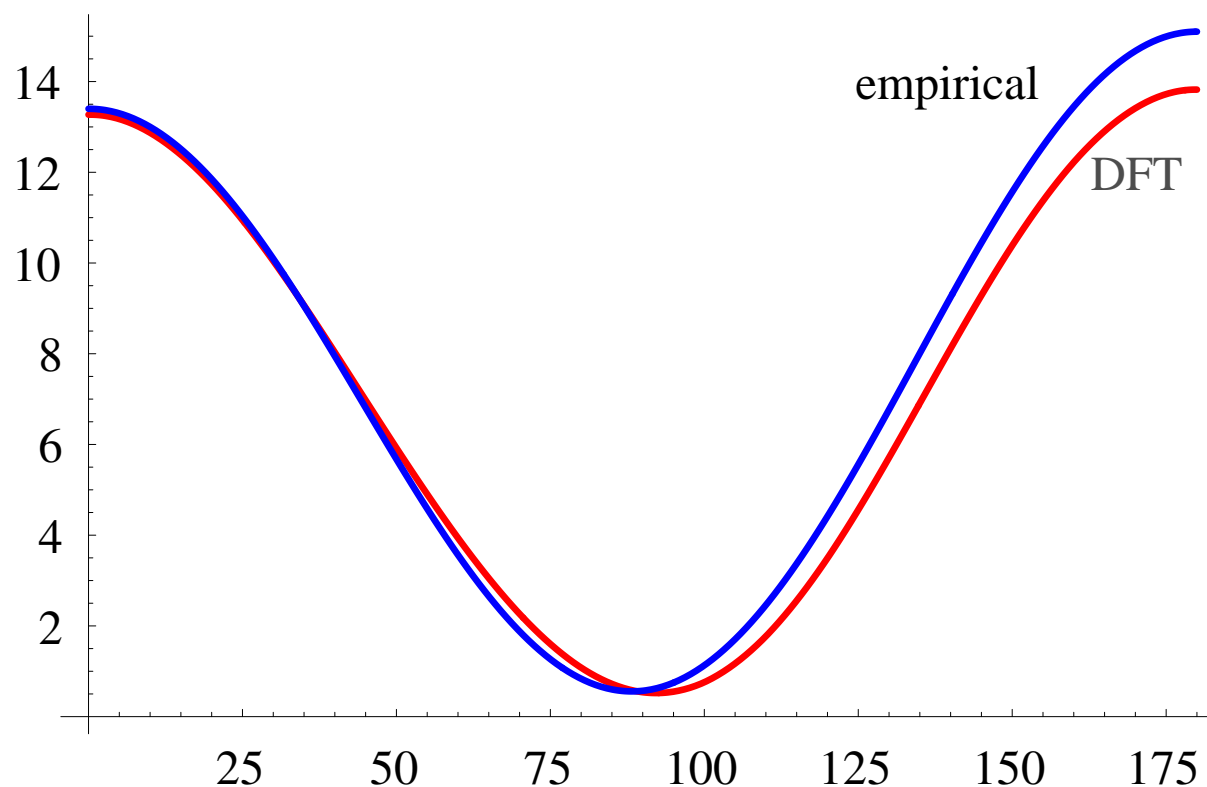
- Accurate spin–spin calculations require large basis sets, augmented with **steep s functions**
 - Huz-III- $su0$: [11s6p2d/6s2p]
 - Huz-III- $su3$: [14s6p2d/9s2p]
- B3LYP calculations on benzene with and without added s functions:

Hz	MCSCF	B3LYP- $su3$	B3LYP- $su0$	emp
$^1 J_{CC}$	70.9	60.0	56.7	56.1
$^2 J_{CC}$	-5.0	-1.8	-1.7	-1.7
$^3 J_{CC}$	19.1	11.2	10.7	9.4
$^1 J_{CH}$	176.7	166.3	151.7	152.7
$^2 J_{CH}$	-7.4	2.0	1.7	1.4
$^3 J_{CH}$	11.7	8.0	7.3	7.0
$^4 J_{CH}$	-1.3	-1.2	-1.3	-1.0
$^3 J_{HH}$		8.7	7.6	7.0
$^4 J_{HH}$		1.3	1.1	1.2
$^5 J_{HH}$		0.8	0.7	0.6

- **Cancellation of errors** gives excellent agreement without added steep functions :)

The Karplus curve

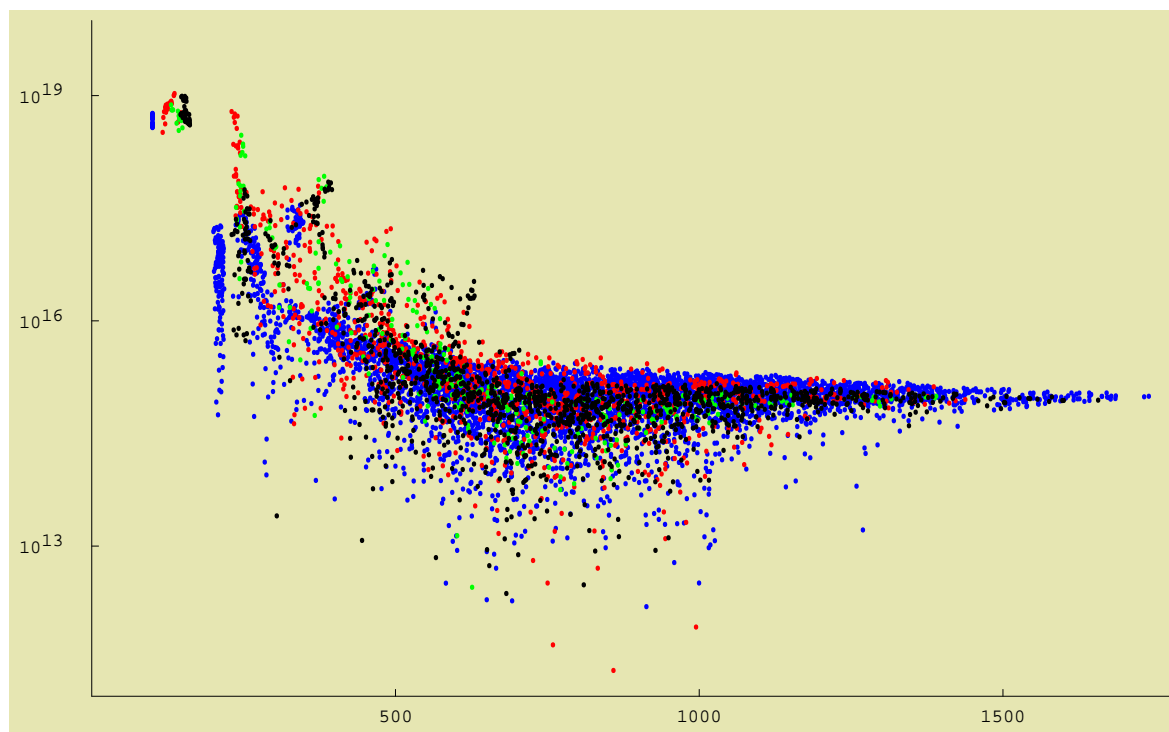
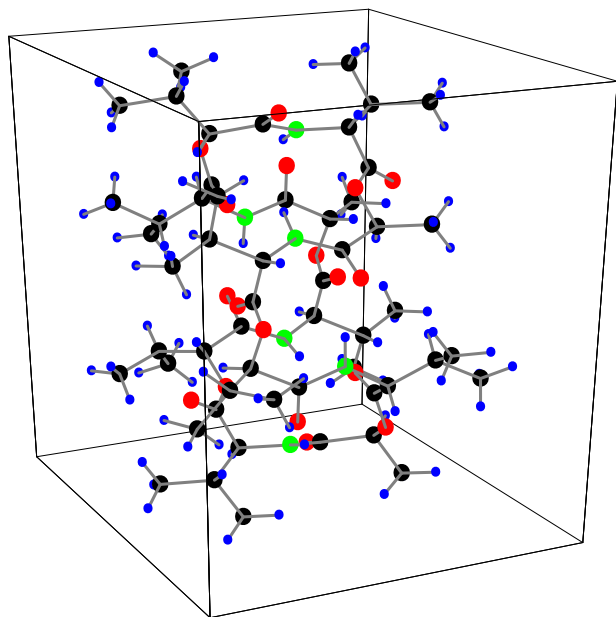
- Vicinal couplings depend critically on the dihedral angle:
- ${}^3J_{\text{HH}}$ in ethane as a function of the dihedral angle:



- The agreement with the empirical Karplus curve is good.

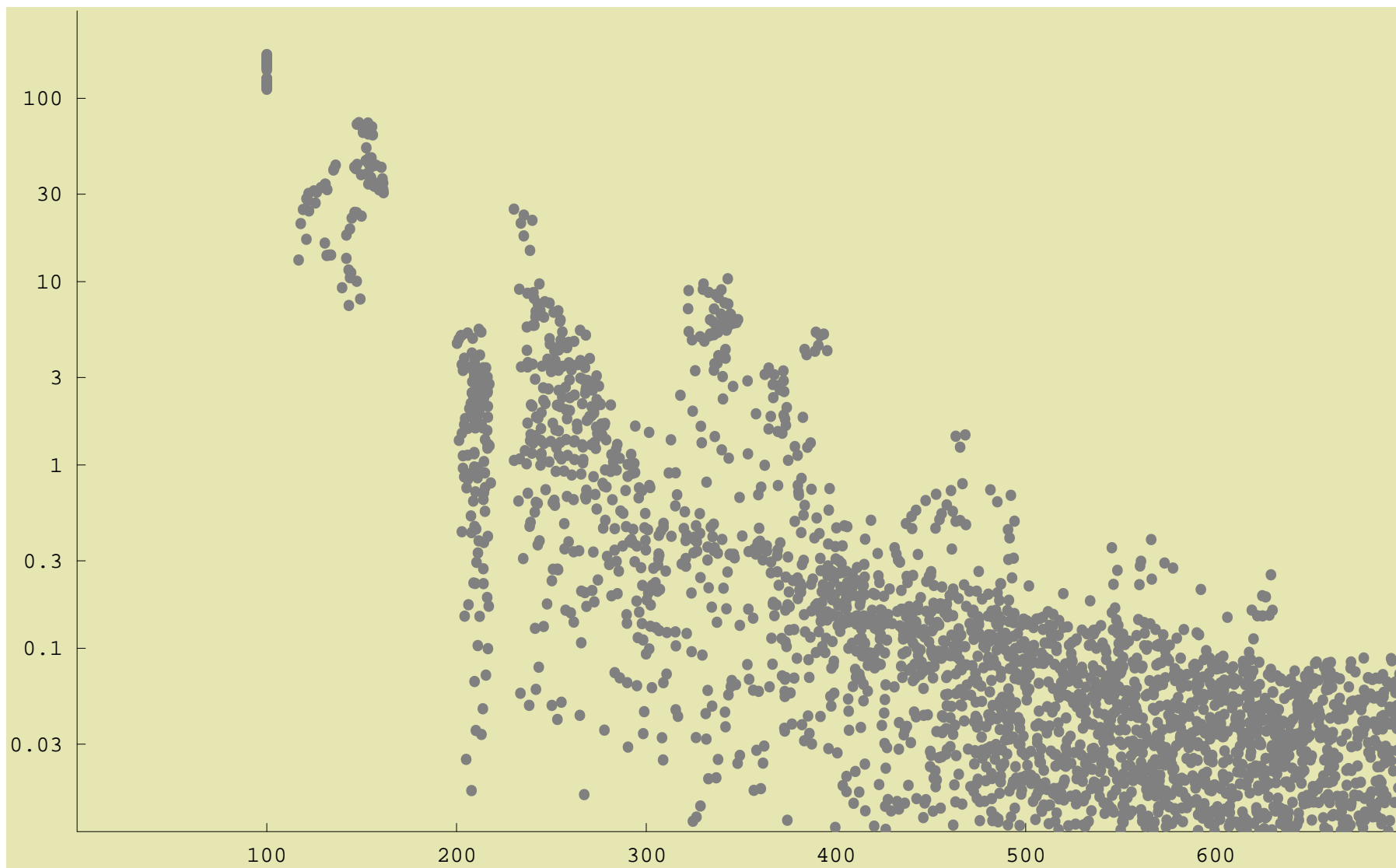
Valinomycin $C_{54}H_{90}N_8O_{18}$

- DFT can be applied to large molecular systems such as valinomycin (168 atoms)
 - there are a total of 7587 spin–spin couplings to the carbon atoms in valinomycin
 - below, we have plotted the magnitude of the reduced LDA/6-31G coupling constants on a logarithmic scale, as a function of the internuclear distance:

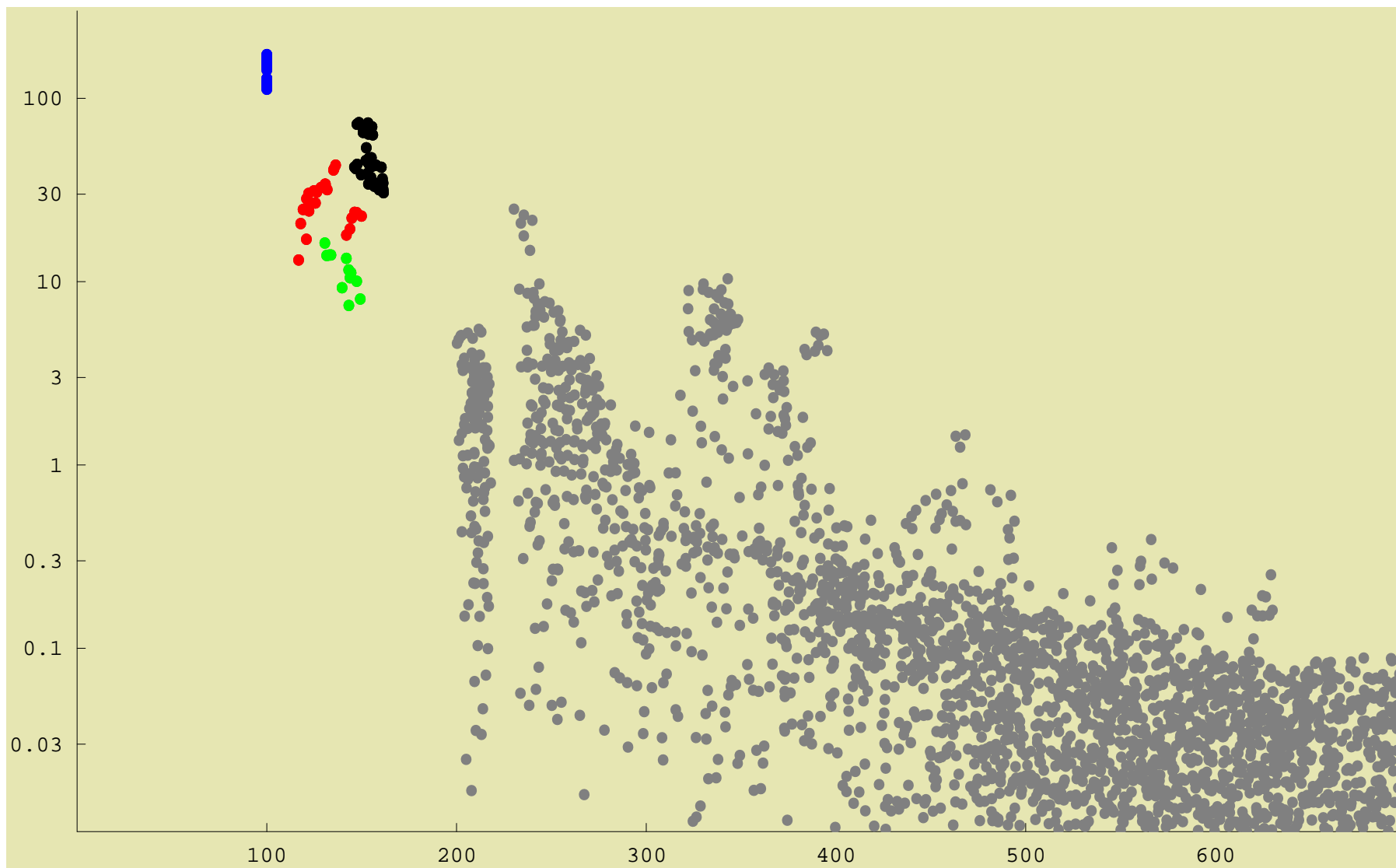


- the coupling constants decay in characteristic fashion, which we shall examine
- most of the indirect couplings beyond 500 pm are small and cannot be detected

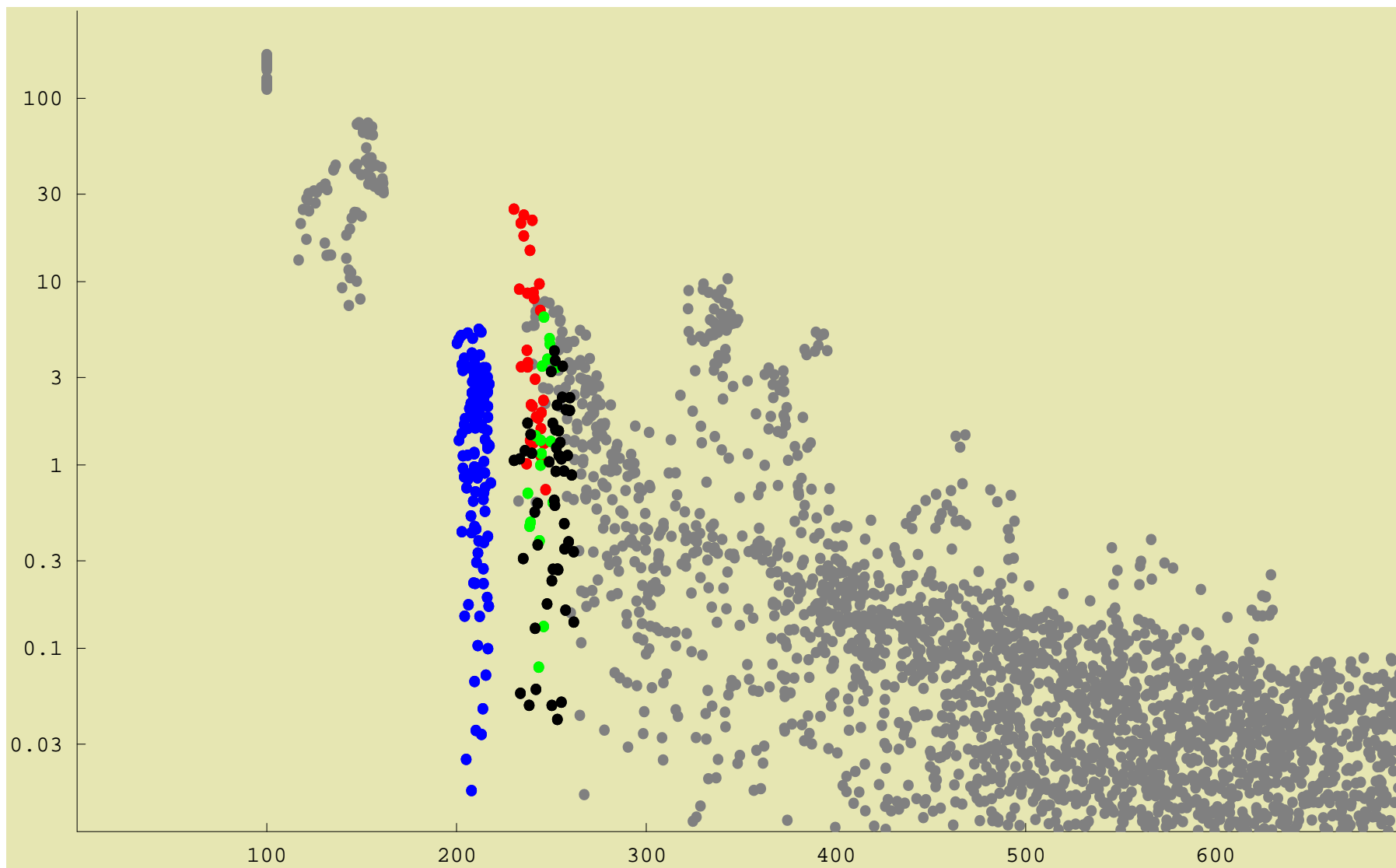
Valinomycin LDA/6-31G spin-spin couplings to CH, CO, CN, CC greater than 0.01 Hz



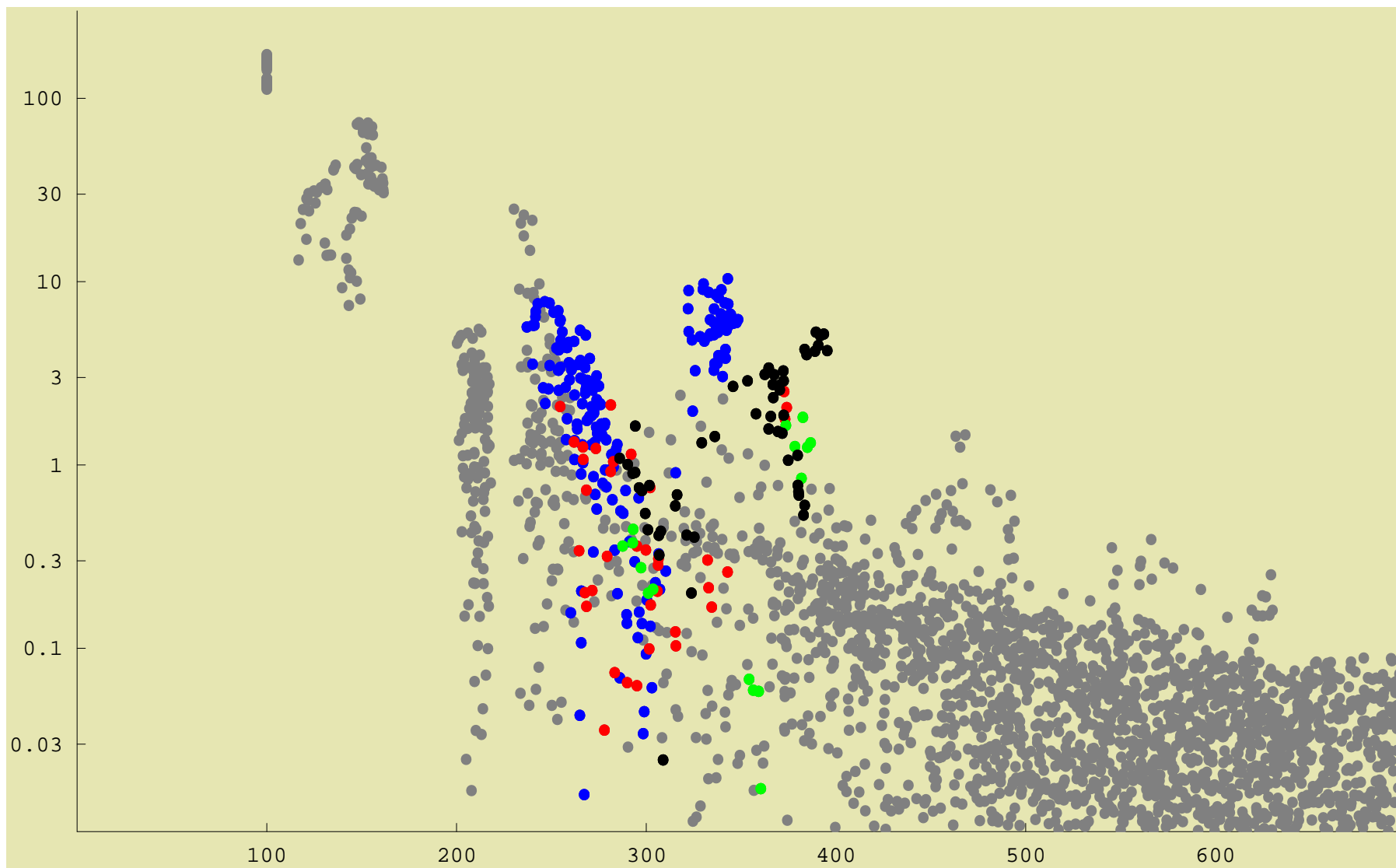
Valinomycin LDA/6-31G spin-spin couplings to CH, CO, CN, CC greater than 0.01 Hz



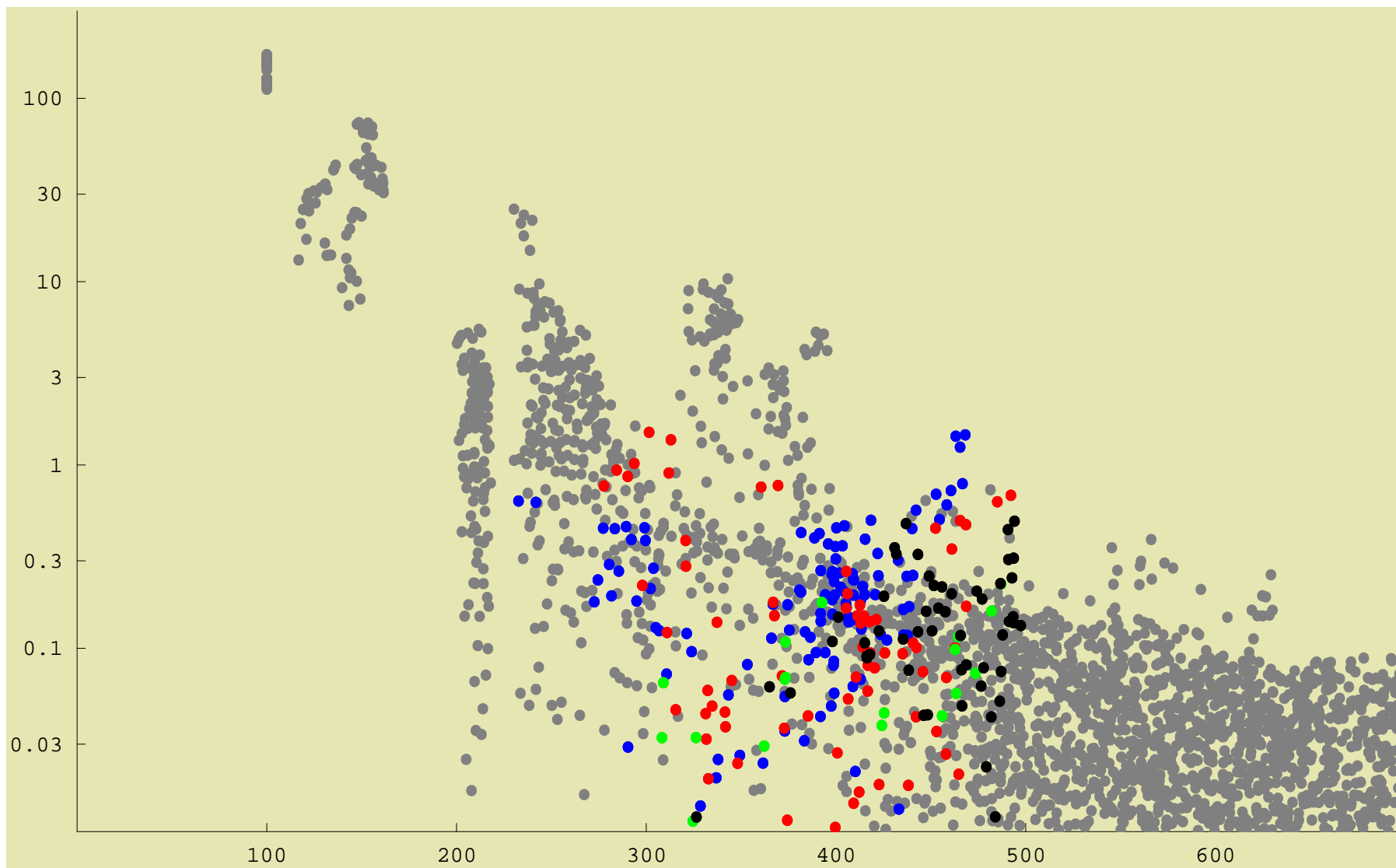
Valinomycin LDA/6-31G spin-spin couplings to CH, CO, CN, CC greater than 0.01 Hz



Valinomycin LDA/6-31G spin-spin couplings to CH, CO, CN, CC greater than 0.01 Hz



Valinomycin LDA/6-31G spin-spin couplings to CH, CO, CN, CC greater than 0.01 Hz

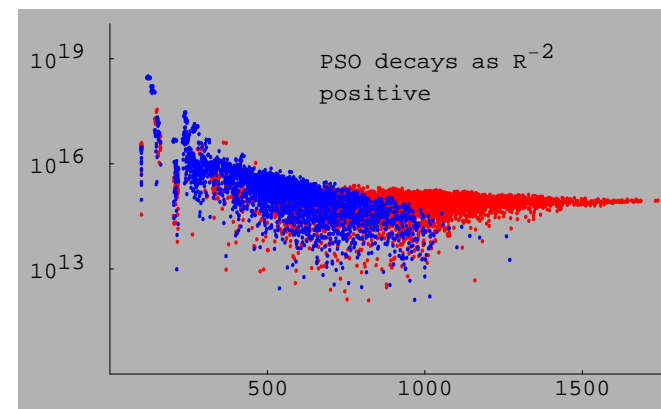
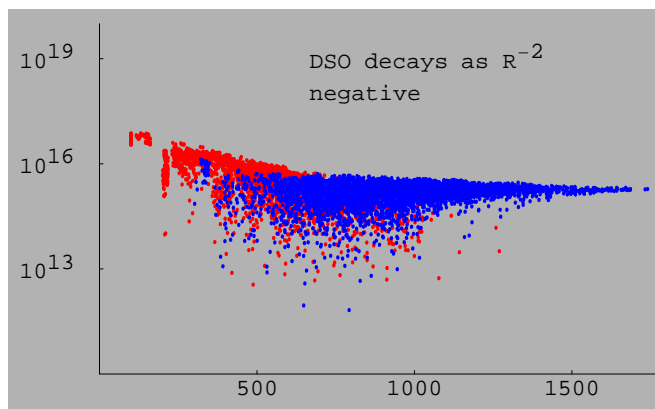
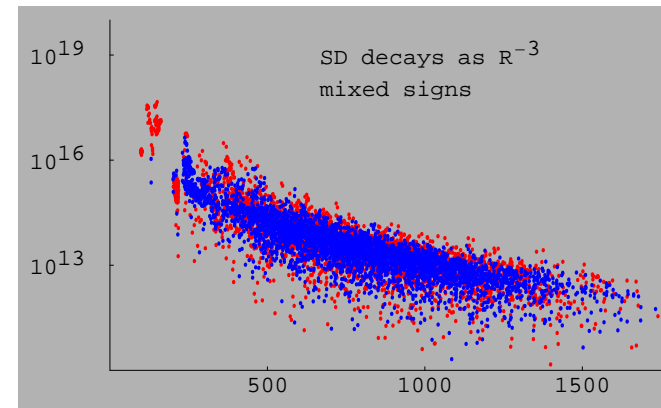
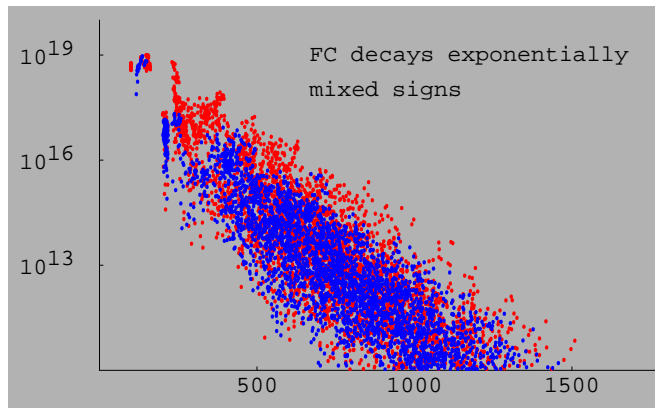


The different long-range decays of the Ramsey terms

- Letting M be the center of the product Gaussian $G_a G_b$, we obtain

$$\begin{aligned} \langle G_a | h_K^{\text{FC}} | G_b \rangle &\propto \exp(-\mu R_{KM}^2), & \langle G_a | h_K^{\text{SD}} | G_b \rangle &\propto R_{KM}^{-3}, \\ \langle G_a | h_{KL}^{\text{DSO}} | G_b \rangle &\propto R_{KM}^{-2} R_{LM}^{-2}, & \langle G_a | h_K^{\text{PSO}} | G_b \rangle &\propto R_{KM}^{-2} \end{aligned}$$

- Insertion in Ramsey's expression gives (red positive, blue negative)



The long-range orbital contributions

- At large separations, the couplings are dominated by the orbital contributions:

$$\mathbf{J}_{KL}^{\text{DSO}} \propto R_{KL}^{-2}, \quad \mathbf{J}_{KL}^{\text{PSO}} \propto R_{KL}^{-2} \quad \leftarrow \text{large separations}$$

- Moreover, in this limit, the **DSO contributions** all become negative:

$$K_{KL}^{\text{DSO}} = \frac{2\alpha^4}{3} \left\langle 0 \left| r_K^{-3} r_L^{-3} \mathbf{r}_K \cdot \mathbf{r}_L \right| 0 \right\rangle < 0 \quad \leftarrow \text{large separations}$$

- Also, the **PSO contributions** become positive, nearly cancelling the DSO contributions.
 - use of Taylor expansion, the virial theorem, and the resolution of identity give:

$$\mathbf{J}_{KL}^{\text{DSO}} + \mathbf{J}_{KL}^{\text{PSO}} \propto R_{KL}^{-3} \quad \leftarrow \text{large separations, large basis}$$

- However, the PSO contributions converge very slowly to the basis-set limit:

